ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United Stated) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-154 Proposed by S. H. L. Kung, Jacksonville University, Jacksonville, Florida

What is special about the following "magic" square?

$$\begin{bmatrix} 11 & 2 & 14 & 19 & 21 \\ 8 & 13 & 3 & 22 & 1 \\ 20 & 17 & 15 & 6 & 9 \\ 7 & 24 & 18 & 10 & 12 \\ 25 & 5 & 23 & 16 & 4 \end{bmatrix}$$

B-155 Composite of Proposals by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada, and Carol Anne Vespe, University of New Mexico, Albuquerque, N. Mex.

Let the n^{th} Pell number be defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that

$$P_{n+a}P_{n+b} - P_{n+a+b}P_n = (-1)^n P_a P_b \,.$$

B-156 Proposed by Allan Scott, Phoenix, Arizona.

Let F_n be the n^{th} Fibonacci number, $G_n = F_{4n} - 2n$, and H_n be the remainder when G_n is divided by 10.

(a) Show that the sequence $H_2, H_3, H_4, \cdots$ is periodic and find the repeating block.

(b) The last two digits of G_9 and G_{14} give Fibonacci numbers 34 and 89 respectively. Are there any other cases?

B-157 Proposed by Klaus Günther Recke, University of Gottingen, Germany.

Let F_n be the n^{th} Fibonacci number and $\{g_n\}$ any sequence. Show that

$$\sum_{k=1}^{n} (g_{k+2} + g_{k+1} - g_k) F_k = g_{n+2} F_n + g_{n+1} F_{n+1} - g_1 .$$

B-158 Proposed by Klaus Günther Recke, University of Gottingen, Germany.

Show that

$$\sum_{k=1}^{n} (kF_k)^2 = \left[(n^2 + n + 2) F_{n+2}^2 - (n^2 + 3n + 2) F_{n+1}^2 - (n^2 + 3n + 4) F_n^2 \right] /2.$$

B-159 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let T_n be the n^{th} triangular number $n(n + 1)/2$ and let $\phi(n)$ be the Euler totient. Show that $\phi(n) \mid \phi(T_n)$ for $n = 1, 2, \cdots$.

SOLUTIONS

NOTE: The name of M. N. S. Swamy was inadvertently omitted from the lists of solvers of B-118, B-119, and B-135.

A PELL ANALOGUE

B-136 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let P_n be the n^{th} Pell number defined by $P_1 = 1$, $P_2 = 2$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that

$$P_{n+1}^2 + P_n^2 = P_{2n+1}.$$

Solution by J. E. Homer, Union Carbide Corporation, Chicago, Ill.

By induction on k it is easily shown that $P_N = P_{k+1}P_{N-k} + P_kP_{N-k-1}$. Letting $N = 2n + 1$ and $k = n$ the desired result follows.

Also solved by Clyde A. Bridger, Timothy Burns, Herta T. Freitag, J. A. H. Hunter (Canada), John Ivie, D. V. Jaiswal (India), Bruce W. King, Douglas Lind, C. B. A. Peck, A. G. Shannon (Australia), M. N. S. Swamy (Canada), Gregory Wulczyn, Michael Yoder, and the proposer.

ANOTHER PELL IDENTITY

B-137 Proposed by Phil Mana, University of New Mexico, Albuquerque, N. Mex.

Let P_n be the n^{th} Pell number. Show that $P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n$.

Solution by Carol Vespe, University of New Mexico, Albuquerque, N. Mex.

Let $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$. Both sides of the identity are of the form

$$c_1(r^2)^n + c_2(rs)^n + c_3(s^2)^n$$

with constant c's. Hence both sides satisfy a recurrence relation

$$y_{n+3} = k_2y_{n+2} + k_1y_{n+1} + k_0y_n,$$

with constant k's. Therefore the identity is proved for all n by the easy verification for $n = 1, 2,$ and 3.

Also solved by Clyde A. Bridger, Herta T. Freitag, J. E. Homer, John Ivie, D. V. Jaiswal (India), Bruce W. King, C. B. A. Peck, A. G. Shannon (Australia), M. N. S. Swamy (Canada), Gregory Wulczyn, Michael Yoder, and the proposer.

B-138 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Show that for any nonnegative integer k and any integer $n > 1$, there is an n-by-n matrix with integral entries whose top row is $F_{k+1}, F_{k+2}, \cdots, F_{k+n}$ and whose determinant is 1.

Solution by J. E. Homer, Union Carbide Corporation, Chicago, Ill.

The g.c.d. of $(F_{k+1}, F_{k+2}, \cdots, F_{k+n})$ is 1. There exists an n-by-n matrix (Problem E1911, American Mathematical Monthly, Aug.-Sept., 1966) with integral entries whose top row is $F_{k+1}, F_{k+2}, \cdots, F_{k+n}$ and whose determinant is the g.c.d. of $(F_{k+1}, F_{k+2}, \cdots, F_{k+n})$.

Solution for $n \geq 4$ by A. C. Shannon, ACER, Hawthorn, Victoria, Australia.

$$\begin{bmatrix} F_{k+1} & F_{k+2} & F_{k+3} & F_{k+4} & \cdots & F_{k+n-2} & F_{k+n-1} & F_{k+n} \\ F_{k+2} & F_{k+3} & 0 & 0 & \cdots & 0 & 0 & F_{k+n-1} \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & F_{k+n-2} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & F_{k+3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & F_{k+3} & F_{k+2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & F_{k+2} & F_{k+1} \end{bmatrix}$$

Also solved by Michael Yoder and the proposer.

B-139 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show that the sequence $1, 1, 1, 1, 4, 4, 9, 9, 25, 25, \cdots$ defined by $a_{2n-1} = a_{2n} = F_n^2$ is complete even if an a_j with $j \leq 6$ is omitted but that the sequence is not complete if an a_j with $j \geq 7$ is omitted.

Composite of solutions by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania, and the proposer.

Let $S_n = a_1 + \cdots + a_n$. Then it is easily seen that $S_{2m} = 2F_mF_{m+1}$ and $S_{2m-1} = F_{2m} = F_{m+1}^2 - F_{m-1}^2$.

J. L. Brown's criterion (Amer. Math. Monthly, Vol. 68, pp. 557-560) states that a nondecreasing sequence of positive integers $b_1, b_2, \cdots$ with $b_1 = 1$ is complete if and only if $b_{n+1} \leq 1 + b_1 + \cdots + b_n$ for $n = 1, 2, \cdots$.

Thus it suffices to show that

$$\text{(A)} \qquad a_{n+1} \le 1 + S_n - a_i \quad \text{for} \quad 1 \le i \le 6 \quad \text{and} \quad n > i$$

and

$$\text{(B)} \qquad a_{n+1} > 1 + S_n - a_i \quad \text{for} \quad i > 6 \quad \text{and some} \quad n \ge i .$$

There is no loss of generality in letting $i = 2k$. Then (B) follows with $n = i = 2k$ since $k \ge 4$, $1 - F_{k-1}^2 \le 1 - 2^2 = -3$, and

$$a_{n+1} = F_{k+1}^2 > 1 + F_{k+1}^2 - F_{k-1}^2 = 1 + S_{2k-1} = 1 + S_{2k} - a_{2k} = 1 + S_n - a_i .$$

One easily checks (A) when $n < 6$. With $n = 2m - 1$ and $m \ge 4$, (A) is clear since $S_n - a_i$ contains $a_{n+1} = a_n$ as a term. With $n = 2m$ and $m \ge 3$, (A) holds if

$$a_{n+1} = F_{m+1}^2 \le 1 + S_n - a_6 = S_n - 3 = 2F_mF_{m+1} - 3$$

or if

$$F_{m+1}(2F_m - F_{m+1}) \ge 3$$

or if

$$F_{m+1}(F_m - F_{m-1}) \ge 3 ,$$

which is true for $m \ge 3$.

B-140 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Show that $F_{ab} > F_aF_b$ if a and b are integers greater than 1.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pa.

$ab > a + b - 1$ for this is true for $a,b = 2$ and differentiation with respect to b with a fixed shows that the l.h.s. increases faster than the r.h.s.

in b (and, by symmetry, in a). Then from

$$F_m F_n + F_{m-1} F_{n-1} = F_{m+n-1}$$

(see Fibonacci Quarterly, Vol. 1, No. 1, p. 66),

$$F_{ab} > F_{a+b-1} = F_a F_b + F_{a-1} F_{b-1} > F_a F_b \; .$$

Also solved by J. E. Homer, D. V. Jaiswal (India), A. C. Shannon (Australia), M. N. S. Swamy (Canada), Michael Yoder, and the proposer.

B-141 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Show that no Fibonacci number F_n nor Lucas number L_n is an even perfect number.

Solution by the proposer.

Recall that an even perfect number greater than 6 must leave a remainder of 1 upon division by 9 and must be a multiple of 4. An even perfect number greater than 28 must be a multiple of 16.

If $F_n \equiv 1 \pmod 9$, then $n \equiv 1, 2, 10, 18,$ or $23 \pmod{24}$; if $16 | F_n$ then $n \equiv 0 \pmod{12}$. These two sets have no common elements.

If $L_n = 1 \pmod 9$, then $n \equiv 1$ or $11 \pmod{24}$. If $4 | L_n$ then $n \equiv 3 \pmod 6$. Again we have an empty intersection.

Problem H-23 asked if there were any triangular Fibonacci numbers beyond 55. If the answer to that question is "no" then the Fibonacci half of the above is immediate.

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