PRODUCTS OF FIBONACCI AND LUCAS NUMBERS

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Let $U_{x_1}$ denote a Fibonacci or a Lucas number and consider the product

$$U_{x_1} U_{x_2} \cdots U_{x_n}.$$ 

We are interested in finding a general method by which this product may be "expanded," i.e., expressed as a linear function of Fibonacci or Lucas numbers.

Beginning with the case in which $n = 2$ we find that there are four types of such products. Using Binet's formulas it is easily verified that these may be expressed as follows:

$$F_{x_1} L_{x_2} = F_{x_1+x_2} + (-1)^{x_2} F_{x_1-x_2}$$

$$L_{x_1} F_{x_2} = F_{x_1+x_2} - (-1)^{x_2} F_{x_1-x_2}$$

$$L_{x_1} L_{x_2} = L_{x_1+x_2} + (-1)^{x_2} L_{x_1-x_2}$$

$$F_{x_1} F_{x_2} = \frac{1}{5} \left[ L_{x_1} L_{x_2} - (-1)^{x_2} L_{x_1-x_2} \right]$$

From these four identities we make the following observations.

This "multiplication" is not commutative.

The product of a mixed pair (i.e., one factor is a Fibonacci number and the other is a Lucas number) is a linear function of Fibonacci numbers. The product of a Fibonacci and Lucas number is a function of Lucas numbers.

The coefficient of the second term is $(-1)^{x_2}$ or $-(-1)^{x_2}$ according as $x_2$ comes from the subscript of a Lucas or a Fibonacci number.

The factor $1/5$ occurs when both numbers in the product are Fibonacci.
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For convenience we denote -1 by \( \epsilon \). Now consider \( \epsilon^{X_1} \) as playing a dual role. As a coefficient of \( L_{x_1} \) or \( F_x \) it has the value \(-1)^{X_1}\). As an operator applied to these numbers it reduces their subscripts by \( 2X_1 \). With this in mind, we may write

\[
F_{X_1} L_{X_2} = (1 + \epsilon^{X_2})F_{X_1+X_2} = F_{X_1+X_2} + (-1)^{X_2}F_{X_1-X_2}
\]

\[
L_{X_1} F_{X_2} = (1 - \epsilon^{X_2})F_{X_1+X_2} = F_{X_1+X_2} - (-1)^{X_2}F_{X_1-X_2}
\]

\[
L_{X_1} L_{X_2} = (1 + \epsilon^{X_2})L_{X_1+X_2} = L_{X_1+X_2} + (-1)^{X_2}L_{X_1-X_2}
\]

\[
F_{X_1} F_{X_2} = (1 - \epsilon^{X_2})L_{X_1+X_2} = \frac{1}{3}[L_{X_1+X_2} - (-1)^{X_2}L_{X_1-X_2}]
\]

We turn now to products containing three factors such as \( L_{X_1} L_{X_2} F_{X_3} \). For the moment we shall understand that \( L_{X_1} L_{X_2} F_{X_3} \) means \( (L_{X_1} L_{X_2}) F_{X_3} \). Then, making use of the above results, we have

\[
(L_{X_1} L_{X_2}) F_{X_3} = \left[ L_{X_1+X_2} + (-1)^{X_2}L_{X_1-X_2} \right] F_{X_3}
\]

\[
= L_{X_1+X_2} F_{X_3} + (-1)^{X_2}L_{X_1-X_2} F_{X_3}
\]

\[
= F_{X_1+X_2+X_3} - (-1)^{X_3} F_{X_1+X_2-X_3} + (-1)^{X_2} \times
\]

\[
\times \left[ F_{X_1-X_2+X_3} - (-1)^{X_3} L_{X_1-X_2-X_3} \right]
\]

\[
= F_{X_1+X_2+X_3} - (-1)^{X_3} F_{X_1+X_2-X_3} + (-1)^{X_2} \times
\]

\[
\times \ F_{X_1-X_2+X_3} - (-1)^{X_3} L_{X_1-X_2-X_3}
\]

Using \( \epsilon^{X_1} \) we arrive at the same result.
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\[
L_{x_1} L_{x_2} F_{x_3} = (1 + \epsilon X^2)(1 - \epsilon X^3)F_{x_1+x_2+x_3}
\]

\[
= (1 + \epsilon X^2)F_{x_1+x_2+x_3} - (-1)^{X^3}F_{x_1+x_2-x_3}
\]

\[
= F_{x_1+x_2+x_3} - (-1)^{X^3}F_{x_1-x_2-x_3} + (-1)^{X^3}F_{x_1-x_2+x_3} - (-1)^{X^2}X^9 \times \n F_{x_1-x_2-x_3}.
\]

Since

\[
(1 + \epsilon X^2)(1 - \epsilon X^3) = 1 + \epsilon X^2 - \epsilon X^3 - \epsilon X^2 X^3,
\]

we could proceed as follows:

\[
L_{x_1} L_{x_2} F_{x_3} = (1 + \epsilon X^2 - \epsilon X^3 - \epsilon X^2 X^3)F_{x_1+x_2+x_3}
\]

\[
= F_{x_1+x_2+x_3} - (-1)^{X^3}F_{x_1-x_2-x_3} + (-1)^{X^3}F_{x_1-x_2+x_3} - (-1)^{X^2}X^9 \times \n F_{x_1-x_2-x_3}.
\]

We leave it as an exercise to show that \(L_{x_1} (L_{x_2} F_{x_3})\) when expanded by any of these methods leads to the same result.

There are eight types of products, each consisting of three factors. We list them below.

\[
F_{x_1} L_{x_2} L_{x_3} = (1 + \epsilon X^2)(1 + \epsilon X^3)F_{x_1+x_2+x_3}
\]

\[
L_{x_1} F_{x_2} L_{x_3} = (1 - \epsilon X^2)(1 + \epsilon X^3)F_{x_1+x_2+x_3}
\]

\[
L_{x_1} L_{x_2} F_{x_3} = (1 + \epsilon X^2)(1 - \epsilon X^3)F_{x_1+x_2+x_3}
\]
The preceding results are the bases for the following conjecture.

Let $U_{x_1}$ represent a Fibonacci or a Lucas number. Let $p$ be the number of Fibonacci numbers in a product of both Fibonacci and Lucas numbers. Let

$$\overline{U}_{x_1 x_2 \cdots x_n}$$

denote a Fibonacci or a Lucas number according as $p$ is odd or even. As a coefficient $e^{x_1}$ has the numerical value $(-1)^{x_1}$ but as an operator applied to

$$\overline{U}_{x_1 x_2 \cdots x_n},$$

it reduces the subscript of the latter by $2x_1$.

Use

$$(1 - e^{x_1}) \quad \text{or} \quad (1 + e^{x_1})$$
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according as $x_i$ is the subscript of a Fibonacci or a Lucas number in the product. Then

$$
\prod_{i=1}^{u} U_{x_i} = \frac{1}{2 \binom{2}{u}} (1 \pm \epsilon^{x_2})(1 \pm \epsilon^{x_3}) \cdots (1 \pm \epsilon^{x_u})U_{\sum x_i^2+\cdots+x_u^2}.
$$

The proof of this conjecture is given at the end of this article. The following example will illustrate

$$
F_{15} F_{12} L_{20} F_{8} = \frac{1}{5} (1 - \epsilon^{12})(1 + \epsilon^{10})(1 - \epsilon^{8})F_{45}
$$

$$
= \frac{1}{5} (1 - \epsilon^{12})(1 + \epsilon^{10})(F_{45} - F_{23})
$$

$$
= \frac{1}{5} (1 - \epsilon^{12})(F_{45} - F_{29} + F_{25} - F_{9})
$$

$$
= \frac{1}{5} (F_{45} - F_{29} + F_{25} - F_{9} - F_{21} + F_{5} - F_{1} + F_{-15})
$$

$$
= \frac{1}{5} (F_{45} - F_{29} + F_{25} - F_{9} + F_{15} - F_{8} + F_{5} - F_{1}) .
$$

The above rule also applies if the product consists entirely of Fibonacci or of Lucas numbers each with the same subscript. For example,

$$
L_5^x = (1 + \epsilon^x)^4 L_5^x
$$

$$
= (1 + 4 \epsilon^x + 6 \epsilon^{2x} + 4 \epsilon^{3x} + \epsilon^{4x}) L_5^x
$$

$$
= L_5^x + 4(-1)^x L_3^x + 6(-1)^{2x} L_{-x} + 4(-1)^{3x} L_{-x} + (-1)^{4x} L_{-3x}
$$

$$
= L_5^x + \left[4(-1)^x + (-1)^x\right] L_3^x + \left[6(-1)^{2x} + 4(-1)^{2x}\right] L_{-x}
$$

$$
= L_5^x + 5(-1)^x L_3^x + 10 L_{-x} .
$$

More generally, if $n$ is an odd integer we have
\[
L^n_x = (1 + e^x)^{n-1} L_{nx}
\]

\[
= L_{nx} + \binom{n-1}{1} e^x L_{(n-2)x} + \binom{n-1}{2} e^{2x} L_{(n-4)x} + \cdots
\]

\[
+ \binom{n-1}{n-2} e^{(n-2)x} L_{-(n-4)x} + \binom{n-1}{n-1} e^{(n-1)x} L_{-(n-2)x}
\]

Since

\[
L_{-k} = (-1)^k L_k,
\]

we get

\[
L^n_x = L_{nx} + \left[ \binom{n-1}{1} + \binom{n-1}{n-1} \right] e^x L_{(n-2)x} + \left[ \binom{n-1}{2} + \binom{n-1}{n-2} \right] e^{2x} L_{(n-4)x} + \cdots + \binom{n-1}{\frac{n-1}{2}} + \binom{n-1}{\frac{n+1}{2}} \right] e^{\frac{n-1}{2}} L_{x}
\]

Making use of the identity

\[
\binom{n}{m} + \binom{n}{n-m} = \binom{n+1}{m},
\]

the last equation may be written

\[
L^n_x = L_{nx} + \binom{n}{1} e^x L_{(n-2)x} + \binom{n}{2} e^{2x} L_{(n-4)x} + \cdots + \binom{n}{\frac{n-1}{2}} e^{\frac{n-1}{2}} L_{x}
\]

\[
L^n_x = \sum_{i=0}^{\frac{n-1}{2}} (-1)^i \binom{n}{i} L_{(n-2i)x} \quad n = 1, 3, 5, \cdots
\]

Similarly, we get the following:
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$$L^n_x = \sum_{i=0}^{n-1} \frac{(-1)^{x_i}}{5} \binom{n}{i} L_{(n-2i)x} + 2\left(\frac{n}{2}\right)^2 \left(\frac{n-1}{2}\right)$$ (n, even)

$$P^n_x = \frac{1}{n-1} \sum_{i=0}^{n-1} \frac{(-1)^{(x+1)i}}{5} \binom{n}{i} P_{(n-2i)x}$$ (n, odd)

$$P^n_x = \frac{1}{n-1} \sum_{i=0}^{n-1} \frac{(-1)^{(x+1)i}}{5} \binom{n}{i} L_{(n-2i)x} + 2\left(\frac{n}{2}\right)^2 \left(\frac{n-1}{2}\right)$$ (n, even)

The proof of the rule which has been used to express products of Fibonacci and Lucas numbers as linear functions of those numbers is a proof by induction.

We have seen that it is true for \( n = 2 \) and \( n = 3 \). Assume it is true for all integral values of \( n \) up to and including \( k \). Then, if \( p \) is even

$$(1) \quad \prod_{i=1}^{k} U_{x_i} = \frac{1}{5^2} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) L_{x_1+x_2+\cdots+x_k}$$

Multiplying both members of this equation by \( L_{x+1} \) we get

$$\prod_{i=1}^{k} U_{x_i} L_{x+1} = \frac{1}{5^2} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) L_{x_1+x_2+\cdots+x_k} L_{x+k+1}$$

$$= \frac{1}{5^2} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k}) \times$$

$$\times (L_{x_1+x_2+\cdots+x_k+1} + (-1)^{k+1} L_{x_1+x_2+\cdots+x_k-x_k+1})$$

$$= \frac{1}{5^2} (1 \pm \epsilon^{x_2}) \cdots (1 \pm \epsilon^{x_k})(1 + \epsilon^{x_k+1}) L_{x_1+x_2+\cdots+x_k+1}$$
Next, multiplying both sides of equation (1) by \( F_{x+1} \) we get

\[
\prod_{i=1}^{k} U_{x_i} F_{x_{k+1}} = \frac{1}{\sqrt{5}} \left( 1 + e^{x_2} \right) \cdots \left( 1 \pm e^{x_k} \right) L_{x_1+x_2+\cdots+x_k} F_{x_{k+1}}
\]

\[
= \frac{1}{\sqrt{5}} \left( 1 \pm e^{x_2} \right) \cdots \left( 1 \pm e^{x_k} \right) \times
\]

\[
\times \left[ F_{x_1+x_2+\cdots+x_{k+1}} - (-1)^{k+1} L_{x_1+x_2+\cdots+x_k} F_{x_{k+1}} \right]
\]

\[
= \frac{1}{\sqrt{5}} \left( 1 \pm e^{x_2} \right) \cdots \left( 1 \pm e^{x_k} \right) \left( 1 - e^{x_{k+1}} \right) F_{x_1+x_2+\cdots+x_{k+1}}
\]

Since both of these results agree with that given by the general rule for \( n = k + 1 \) the induction is complete for the case in which

\[
\bar{U}_{x_1+x_2+\cdots+x_n} = L_{x_1+x_2+\cdots+x_n}
\]

We leave the case in which

\[
\bar{U}_{x_1+x_2+\cdots+x_n} = F_{x_1+x_2+\cdots+x_n}
\]

for the reader to prove.

We now consider the reverse problem; that is, the problem of finding a general method of expressing

\[
L_{x_1+x_2+\cdots+x_n} \quad \text{and} \quad F_{x_1+x_2+\cdots+x_n}
\]

as a homogeneous function of products, each of the type,

\[
F_{x_1} F_{x_2} \cdots F_{x_i} L_{x_{i+1}} L_{x_{i+2}} \cdots L_{x_n}
\]
For simplicity let $S^n_i$ denote the sum of all products consisting of $i$ factors which are Fibonacci numbers and $n-i$ which are Lucas numbers. The number of such factors is, of course, $\binom{n}{i}$.

For example,

$$S^n_4 = F_{x_1} F_{x_2} L_{x_3} L_{x_4} + F_{x_1} F_{x_3} L_{x_2} L_{x_4} + F_{x_1} F_{x_4} L_{x_2} L_{x_3} + F_{x_2} F_{x_3} L_{x_1} L_{x_4} + F_{x_2} F_{x_4} L_{x_1} L_{x_3} + F_{x_3} F_{x_4} L_{x_1} L_{x_2}.$$  

For later use we note that

$$S^n_1 L_{x_{n+1}} + S^n_{i-1} F_{x_{n+1}} = S^n_{i+1}.$$  

This follows from the identity

$$\binom{n}{i} + \binom{n}{i-1} = \binom{n+1}{i}.$$  

For the case $n = 2$ we readily prove (using Binet's formulas) that

$$F_{x_1 + x_2} = \frac{1}{2} (L_{x_1} F_{x_2} + F_{x_1} L_{x_2}) = \frac{1}{2} S^n_1.$$  

$$L_{x_1 + x_2} = \frac{1}{2} (L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2}) = \frac{1}{2} (S^n_0 + 5 S^n_2).$$  

Using these two identities as a basis, we develop the following for $n = 3$
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\[ F_{x_1+x_2+x_3} = F_{(x_1+x_2)+x_3} \]
\[ = \frac{1}{2} \left[ L_{x_1+x_2} F_{x_3} + F_{x_1+x_2} L_{x_3} \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} \left( L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2} \right) F_{x_3} + \frac{1}{2} \left( L_{x_1} F_{x_2} + F_{x_1} L_{x_2} \right) L_{x_3} \right] \]
\[ = \frac{1}{2} \left[ L_{x_1} L_{x_2} F_{x_3} + 5 F_{x_1} F_{x_2} \right] L_{x_3} + \frac{5}{2} \left( L_{x_1} F_{x_2} + F_{x_1} L_{x_2} \right) L_{x_3} \]
\[ = \frac{1}{2^2} \left[ S_3^4 + 5 S_3^3 \right] \]

\[ L_{x_1+x_2+x_3} = L_{(x_1+x_2)+x_3} \]
\[ = \frac{1}{2} \left[ L_{x_1+x_2} L_{x_3} + 5 F_{x_1+x_2} F_{x_3} \right] \]
\[ = \frac{1}{2} \left[ \frac{1}{2} \left( L_{x_1} L_{x_2} + 5 F_{x_1} F_{x_2} \right) L_{x_3} + \frac{5}{2} \left( L_{x_1} F_{x_2} + F_{x_1} L_{x_2} \right) L_{x_3} \right] \]
\[ = \frac{1}{2} \left[ L_{x_1} L_{x_2} L_{x_3} + 5 F_{x_1} F_{x_2} \right] L_{x_3} + \frac{5}{2} \left( L_{x_1} F_{x_2} + F_{x_1} L_{x_2} \right) L_{x_3} \]
\[ = \frac{1}{2^2} \left[ S_3^5 + 5 S_3^4 \right] \]

Proceeding in this manner we derive the following identities for \( n = 4 \) and \( n = 5 \):

\[ F_{x_1+x_2+x_3+x_4} = \frac{1}{2^3} \left[ S_4^4 + 5 S_4^3 \right] \]
\[ F_{x_1+x_2+x_3+x_4+x_5} = \frac{1}{2^4} \left[ S_5^5 + 5 S_5^4 + 5^2 S_5^3 \right] \]
\[ L_{x_1+x_2+x_3+x_4} = \frac{1}{2^4} \left[ S_4^5 + 5 S_4^4 + 5^2 S_4^3 \right] \]
\[
L_{x_1+x_2+\cdots+x_5} = \frac{1}{2^4} \left[ S_5^5 + 5 S_2^5 + 5^2 S_4^5 \right].
\]

From the above results we conjecture the validity of the following identities which we will prove later.

\[
(2) \quad F_{x_1+x_2+\cdots+x_n} = \frac{1}{2^{n-1}} \left[ S_1^n + 5 S_2^n + 5^2 S_3^n + \cdots + \left( \frac{n-2}{5} \right) S_{n-1}^n \right] \quad (n, \text{ even})
\]

\[
(3) \quad L_{x_1+x_2+\cdots+x_n} = \frac{1}{2^{n-1}} \left[ S_0^n + 5 S_2^n + 5^2 S_4^n + \cdots + \left( \frac{n-1}{5} \right) S_{n-1}^n \right] \quad (n, \text{ odd}).
\]

Before proceeding with the proofs of these identities we consider the special case when \( x_1 = x_2 = \cdots = x_n = x \). For this case we get the following:

\[
F_{nx} = \frac{1}{2^{n-1}} \left[ \begin{array}{c}
(1) F_x L_x^{n-1} + 5 (2) F_x^3 L_x^{n-3} + \cdots + \left( \frac{n-2}{5} \right) F_x^{n-2} L_x^n \\
\frac{n-1}{5} F_x^n \end{array} \right] \quad (n, \text{ even})
\]

\[
L_{nx} = \frac{1}{2^{n-1}} \left[ \begin{array}{c}
L_x^n + 5 (3) F_x^3 L_x^{n-2} + \cdots + \left( \frac{n}{5} \right) F_x^n L_x^n \\
\frac{n-1}{5} F_x^{n-1} L_x^n \end{array} \right] \quad (n, \text{ odd}).
\]

Note, in particular, if \( n = 2 \) we get two well-known identities

\[
F_{2x} = F_x L_x
\]

and

\[
L_{2x} = \frac{1}{2} \left( L_x^2 + 5 F_x^2 \right).
\]
We have now to prove the identities (1) and (2). The proof is by induction on \( n \). Both identities are true for \( n = 2 \). We assume they are valid for all integral values of \( n \) up to and including \( n = k \).

Then

\[
(4) \quad F_{x_1+x_2+\cdots+x_k} = \frac{1}{2^{k-1}} \left[ s_1^k + s_2^k + s_3^k + \cdots + \left( \frac{k-2}{5} s_{k-1}^k \right) \right] (k, \text{ even})
\]

\[
(5) \quad L_{x_1+x_2+\cdots+x_k} = \frac{1}{2^{k-1}} \left[ s_0^k + s_2^k + s_4^k + \cdots + \left( \frac{k-1}{5} s_{k-1}^k \right) \right] (k, \text{ odd})
\]

Now

\[
(6) \quad L_{x_1+x_2+\cdots+x_k+x_{k+1}} = L_{(x_1+x_2+\cdots+x_k)+x_{k+1}}
\]

\[
= \frac{1}{2} \left[ L_{x_1+x_2+\cdots+x_k} L_{x_{k+1}} + 5 F_{x_1+x_2+\cdots+x_k} F_{x_{k+1}} \right].
\]

Applying (4) and (5) to the right member of (6), we get

\[
(7) \quad L_{x_1+x_2+\cdots+x_k} L_{x_{k+1}} = \frac{1}{2^{k-1}} \left[ s_0^k L_{x_{k+1}} + s_2^k L_{x_{k+1}} + \cdots + \left( \frac{k}{5} s_k^k L_{x_{k+1}} \right) \right] (k, \text{ even})
\]

\[
+ \left[ \frac{k-1}{5} s_{k-1}^k L_{x_{k+1}} \right] (k, \text{ odd})
\]
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\[ F_{x_1+x_2+\cdots+x_k} = \frac{1}{2^{k+1}} \left[ k \sum_{i=0}^{k} F_{x_{k+1}} + 5 \sum_{i=0}^{k} F_{x_{k+1}} \right] \]

Substituting in (6) from (7) and (8) and regrouping we get the following:

\[ L_{x_1+x_2+\cdots+x_k} = S_0^{k+1} + 5 \left( S_2^k L_{x_{k+1}} + S_4^k F_{x_{k+1}} \right) \]

Hence

\[ L_{x_1+x_2+\cdots+x_k} = S_0^{k+1} + 5 S_2^{k+1} + 5^2 S_4^{k+1} + \cdots + \left( \frac{k}{5^2} S_k^{k+1} (k+1, \text{ even}) \right) \]

This completes the proof of (3). The proof of (2) is similar.

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ERRATA FOR PSEUDO-FIBONACCI NUMBERS

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Please make the following changes in the above-entitled article appearing in Vol. 6, No. 6:

p. 305: in Eq. (3), O_{i+1} should read: O_{i+2}. On p. 306, the 6th line from the bottom: B^{-k+1} should read: B^{-k+1}. On page 310, in Eq. (12), 2O_{2n} should read: 2\lambda O_{2n}; in Eq. (13), 3O_{2n+1} should read: 3O_{2n+1}. Equation (17), on p. 312: (\lambda - 2)O_{2n-1} should read: \lambda(\lambda - 2)O_{2n-1}. Equation (18) on p. 313: 4O_1 should read: 4O_1. In line 3, p. 314, 2O_{2n+2} should read 2O_{2n+2}, and Eq. (20), p. 315: (\lambda - 2)O_{2n} should read \lambda(\lambda - 2)O_{2n}.

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