IDENTITIES INVOLVING GENERALIZED FIBONACCI NUMBERS

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I. INTRODUCTION

K. Subba Rao [4], and more recently V. C. Harris [1] have obtained some identities involving Fibonacci Numbers F_n defined by

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2} \qquad n \geq 3 .$$

Our object in this paper is to obtain similar results for the generalized Fibonacci Numbers H_n as defined by A. F. Horadam [2],

$$H_1 = p, \quad H_2 = p + q$$

and

$$H_n = H_{n-1} + H_{n-2} \qquad n \geq 3.$$

The numbers p and q are arbitrary. By solving the difference equation for H_n by the usual procedure it is easy to see that

$$H_n = \frac{1}{2\sqrt{5}} \left[l a^n - m b^n \right] \qquad [3]$$

where

$$l = 2(p - qb), \quad m = 2(p - qa)$$

and a and b are the roots of the quadratic equation $x^2 - x - 1 = 0$. We call

$$a = \frac{1 + \sqrt{5}}{2} ; \qquad b = \frac{1 - \sqrt{5}}{2}$$

so that

$$a + b = 1, \quad ab = -1, \quad a - b = \sqrt{5} .$$

By making use of these results we get

$$1 + m = 2(2p - q), \quad 1 - m = 2q\sqrt{5} ,$$

$$\tfrac{1}{4}\, lm = p^2 - pq - q^2 = e \text{ (say)}.$$

It is also easy to see that $H_n = pF_n \neq qF_{n-1}$ where F_n is the n^{th} Fibonacci number given by

$$\frac{a^n - b^n}{\sqrt{5}} .$$

SECTION 2

In this section we obtain certain identities for the generalized Fibonacci numbers. From result (9) of [2] we have the identity

$$H_{r-1}^2 + H_r^2 = (2p - q)\, H_{2r-1} - e\, F_{2r-1} .$$

In this relation putting $r = 2, 3, \cdots, n$ in succession, adding and simplifying, we arrive at the result

$$\sum_{r=1}^{n} H_r^2 = F_n\,[(p + 2q)H_n + eF_{n-1}] + pq\,[(-1)^n - 1] . \tag{1}$$

Consider now $H_{2r-1} = pF_{2r-1} + qF_{2r-2}$ so that

$$\sum_{r=1}^{n} H_{2r-1} = p\sum_{r=1}^{n} F_{2r-1} + q\sum_{r=1}^{n} F_{2r-2} .$$

From the formula for F_n this sum reduces to

(2) $$\sum_{r=1}^{n} H_{2r-1} = H_{2n} - H_2 + H_1$$

(3) $$\sum_{r=1}^{n} H_{2r} = H_{2n+1} - H_1$$

On the same lines we get the following identities

(4) $$\sum_{r=1}^{n} H_{3r-2} = \frac{1}{2}\left[H_{3n} - H_2 + H_1\right]$$

(5) $$\sum_{r=1}^{n} H_{3r-1} = \frac{1}{2}\left[H_{3n+1} - H_1\right]$$

(6) $$\sum_{r=1}^{n} H_{3r} = \frac{1}{2}\left[H_{3n+2} - H_2\right]$$

(7) $$\sum_{r=1}^{n} H_{4r-3} = F_{2n-1}H_{2n} - H_2 + H_1$$

(8) $$\sum_{r=1}^{n} H_{4r-2} = F_{2n}H_{2n}$$

(9) $$\sum_{r=1}^{n} H_{4r-1} = F_{2n}H_{2n+1}$$

(10) $$\sum_{r=1}^{n} H_{4r} = F_{2n+1}H_{2n+1} - H_1$$

(11) $$\sum_{r=1}^{n} H_{2r-1}^2 = \frac{1}{5}\left[H_{2n}(H_{2n-1} + H_{2n+1}) + 2ne + q(q - 2p)\right]$$

(12) $$\sum_{r=1}^{n} H_{2r}^2 = \frac{1}{5}\left[H_{2n+1}(H_{2n} + H_{2n+2}) - 2ne - p(p + 2q)\right]$$

Let us now consider product terms as follows:

(13) $$\sum_{r=1}^{n} H_{2r-2}H_{2r-1} = \frac{1}{5}\left[H_{2n-1}^2 + H_{2n}^2 - ne - (p + q)(p + 2q)\right]$$

(14) $$\sum_{r=1}^{n} H_{2r-1}H_{2r} = \frac{1}{5}\left[H_{2n}^2 + H_{2n+1}^2 + ne - (p^2 + q^2)\right]$$

(15) $$\sum_{r=1}^{n} H_{2r-1}H_{2r+1} = \frac{1}{5}\left[H_{2n+1}(H_{2n} + H_{2n+2}) + 3ne - p(p + 2q)\right]$$

(16) $$\sum_{r=1}^{n} H_{2r}H_{2r+2} = \frac{1}{5}\left[H_{2n+2}(H_{2n+1} + H_{2n+3}) - 3ne - (p + q)(3p + q)\right]$$

Corresponding to the identity

$$F_r^2 - F_{r-k}F_{r+k} = (-1)^{r-k}F_k^2$$

for the generalized Fibonacci numbers we get in the generalized Fibonacci numbers the identity·

(17) $$H_r^2 - H_{r-k}H_{r+k} = (-1)^{r-k}eF_k^2$$

Now consider the sums

(18) $$\sum_{r=1}^{n} H_{2r-2}H_{2r+2} = \frac{1}{5}\left[H_{2n+1}(H_{2n} + H_{2n+2}) - 7ne - (p^2 + 2pq + 10q^2)\right]$$

(19) $$\sum_{r=1}^{n} H_{2r-1}H_{2r+3} = \frac{1}{5}\left[H_{2n+2}(H_{2n+1} + H_{2n+3}) + 7ne - (p+q)(3p+q)\right]$$

Evaluating the quantity $H_kH_{k+1}H_{k+2}$ we get

(20) $$H_kH_{k+1}H_{k+2} = H_{k+1}^3 + (-1)^{k-1}\, eH_{k+1}$$

Therefore

$$H_{2r-1}\,H_{2r}\,H_{2r+1} = H_{2r}^3 + eH_{2r}$$

Hence

$$\sum_{r=1}^{n} H_{2r-1}\,H_{2r}\,H_{2r+1} = \sum_{r=1}^{n} H_{2r}^3 + e\sum_{r=1}^{n} H_{2r}\,.$$

After simplification this becomes,

(21) $$\sum_{r=1}^{n} H_{2r-1}H_{2r}H_{2r+1} = \frac{1}{4}\left[(H_{2n+1}^3 - H_1^3) + e(H_{2n+1} - H_1)\right]$$

(22) $$\sum_{r=1}^{n} H_r^3 = \frac{1}{2}\left[(H_nH_{n+1}^2 - q^2H_2) + e\left\{(p-2q)-(-1)^nH_{n-1}\right\}\right]$$

Now

$$H_{2r}^3 = (pF_{2r} + qF_{2r-1})^3 .$$

On expanding the right side, taking the sum from $r = 1$ to n and simplifying we get the relation

(23) $$\sum_{r=1}^{n} H_{2r}^3 = \frac{1}{4}\,[(H_{2n+1}^3 - H_1^3) - 3e(H_{2n+1} - H_1)]$$

(24) $$\sum_{r=1}^{n} H_{2r}^2 H_{2r-1} = \frac{1}{4}\,[(H_{2n}H_{2n+1}^2 - q^2H_2) + e(H_{2n-1} - H_1)]$$

(25) $$\sum_{r=1}^{n} H_{2r}H_{2r-1}^2 = \frac{1}{4}\,[(H_{2n-1}H_{2n+1}^2 - H_1q^2) + e(H_{2n} - H_2)]$$

(26) $$\sum_{r=1}^{n} H_{2r-1}^2 = \frac{1}{4}\,[(H_{2n}^3 - q^3) + 3e(H_{2n} - q)]$$

From the formula for H_r we can find the sums of the following:

(27) $$\sum_{r=0}^{n} rH_r = nH_{n+2} - H_{n+3} + H_3$$

(28) $$\sum_{r=0}^{n} (-1)^r rH_r = [(-1)^n\left[(n+1)H_{n-1} - H_{n-2}\right] + (3q - 2p)]$$

(29) $$\sum_{r=0}^{n} (-1)^r H_{2r} = \frac{1}{5}\,[(-1)^{n+1}(H_{2n} + H_{2n+2}) - (p + 2q)]$$

(30) $$\sum_{r=0}^{n} (-1)^r H_{2r+1} = \frac{1}{5}\left[(-1)^n (H_{2n+1} + H_{2n+3}) + (2p - q)\right]$$

(31) $$\sum_{r=0}^{n} rH_{2r} = \left[nH_{2n+1} - H_1\right] - \left[H_{2n} - H_2\right]$$

(32) $$\sum_{r=0}^{n} rH_{2r+1} = nH_{2n+2} - \left[H_{2n+1} - H_1\right]$$

(33) $$\sum_{r=0}^{n} (-1)^r rH_{2r} = \frac{1}{5}\left[(-1)^n((n+1)H_{2n} + nH_{2n+2}) - (H_2 - H_1)\right]$$

(34) $$\sum_{r=0}^{n} (-1)^r rH_{2r+1} = \frac{1}{5}\left[(-1)^n((n+1)H_{2n+1} + nH_{2n+3}) - H_1\right]$$

It is easy to see that the list of identities given by K. Subba Rao can be extended to Fibonacci Quaternions defined by

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} .$$

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