SUMS INVOLVING FIBONACCI NUMBERS

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1. INTRODUCTION

In [1] Professor Horadam has defined a certain generalized sequence

$$\{w_n\} \equiv \{w_n(a, b; p, q)\} : \; w_0 = a, \quad w_1 = b$$

and

$$w_n = pw_{n-1} - qw_{n-2} \qquad (n \geq 2)$$

for arbitrary integers a and b. The n^{th} term of this sequence satisfies a relation of the form:

$$w_n = A\alpha^n + B\beta^n$$

where

$$A = \frac{b - a\beta}{\alpha - \beta}; \qquad B = \frac{a\alpha - b}{\alpha - \beta},$$

α and β being the roots of the equation $x^2 - px + q = 0$. He also mentions the particular cases of $\{w_n\}$ given by

$$\begin{aligned}
w_n(1, p; p, q) &= u_n(p, q) \\
w_n(2, p; p, q) &= v_n(p, q) \\
w_n(r, r+s; 1, -1) &= h_n(r, s) \\
w_n(1, 1; 1, -1) &= f_n = u_n(1, -1) = h_n(1, 0) \\
w_n(2, 1; 1, -1) &= l_n = v_n(1, -1) = h_2(2, -1)
\end{aligned}$$

wherein F_n and L_n are the famous Fibonacci and Lucas sequences respectively.

SECTION 2

In this paper our object is to derive some relations connecting the sums of the above sequences up to n terms.

We shall derive a formula for the sum of the most general sequence $\{w_n\}$ and thereby obtain the sums of the other sequences.

Theorem:

$$\sum_{r=0}^{n} w_r = a + \frac{bT_n - aqT_{n-1}}{1 - p + q}$$

where

$$T_n = 1 - \lambda_n ,$$

and

$$\lambda_n = u_n - qu_{n-1} .$$

Consider

$$\sum_{r=0}^{n} w_r = A \sum_{r=0}^{n} \alpha^r + B \sum_{r=0}^{n} \beta^r$$

$$= \frac{b - a\beta}{\alpha - \beta} \frac{\alpha^{n+1} - 1}{\alpha - 1} + \frac{a\alpha - b}{\alpha - \beta} \frac{\beta^{n+1} - 1}{\beta - 1} .$$

This becomes, after simplification by using the facts $(\alpha + \beta) = p$, $\alpha\beta = q$, $\alpha - \beta = d$

$$[(a + b - ap) + aq(u_{n-1} - qu_{n-2}) - b(u_n - qu_{n-1})]/(1 - p + q)$$

Set

$$u_n - qu_{n-1} = \lambda_n .$$

Then, this becomes

$$[(a + b - ap) + aq\lambda_{n-1} - b\lambda_n]/(1 - p + q)$$

$$[a(1 - p + q - q + q\lambda_{n-1}) + b(1 - \lambda_n)]/(1 - p + q)$$

$$a + [-aq(1 - \lambda_{n-1}) + b(1 - \lambda_n)]/(1 - p + q)$$

let now

$$1 - \lambda_n = T_n ,$$

therefore we finally obtain

$$\text{(1)} \qquad \sum_{r=0}^{n} w_r = a + \frac{bT_n - aqT_{n-1}}{1 - p + q} + \cdots$$

Hence the result.

From this we can obtain immediately the sums of Σu_r, Σv_r, ΣF_r, ΣL_r, etc.

$$\sum_{r=0}^{n} u_r (p, q)$$

is obtained by letting $a = 1$, $b = p$ in (1)

$$\text{(2)} \qquad \begin{aligned} \sum_{r=0}^{n} u_r(p, q) &= 1 + \frac{pT_n - qT_{n-1}}{1 - p + q} \\ \sum_{r=0}^{n} u_r(p, q) &= T_{n+1}/(1 - p + q) \cdots \end{aligned}$$

$$\sum_{r=0}^{n} v_n\,(p,\ q)$$

can be obtained by putting $a = 2,\ \ b = p, p, q$ in (1)

(3)

$$\sum_{r=0}^{n} v_r(p, q) = 2 + \frac{p\,T_{n-2q}\,T_{n-1}}{1 - p + q}$$

$$\sum_{r=0}^{n} v_r(p, q) = 1 + \frac{T_{n+1} - q\,T_{n-1}}{1 - p + q} \quad \cdots$$

In particular,

$$\Sigma w_r(1, 1;\ 1, -1) = \Sigma F_r = \Sigma u_r(1, -1) = \Sigma h_r(1, 0)$$

and

$$\Sigma\, w_r(2, 1;\ 1, -1) = \Sigma L_r = \Sigma v_r(1, -1) = \Sigma h_r(2, -1).$$

(i)

$$\sum_{r=0}^{n} u_r(1,\ -1)$$

is derived by putting $a = b = p = 1,\ \ q = -1$ in (1).

In this case $\lambda_n = u_n + u_{n-1} = u_{n+1}$. Therefore

$$\sum_{r=0}^{n} u_r(1, -1) = 1 + \frac{(1 - u_{n+1}) + (1 - u_n)}{1 - 1 - 1}$$

$$= 1 - [(1 - u_{n+1}) + (1 - u_n)]$$

$$\sum_{r=0}^{n} u_r(1, -1) = u_{n+2} - 1 = F_{n+2} - 1 \qquad [3] \cdots (1_i)$$

This can be verified for any n.

(ii) To get $\Sigma v_r(1, -1)$ let $a = 2$, $b = p = 1$, $q = -1$ in (1). Here also $\lambda_n = u_{n+1}$. So

$$\sum_{r=0}^{n} v_r(1, -1) = 2 + \frac{(1 - u_{n+1}) + 2(1 - u_n)}{1 - 1 - 1}$$

$$= 2 - [3 - 2u_n - u_{n+1}]$$

$$= u_n + u_{n+2} - 1$$

$$= v_{n+2} - 1 \qquad \cdots (1_{ii})$$

This also can be very easily verified for any n.

(iii) Now to evaluate

$$\sum_{r=0}^{n} h_r(p, q) ,$$

set

$$a = p, \quad b = p + q, \quad p = 1, \quad q = -1$$

in (1). Here again

$$\lambda_n = u_{n+1} = F_{n+1} .$$

Then

$$\sum_{r=0}^{n} h_r(p,q) = p - [(p+q)(1 - F_{n+1}) + p(1 - F_n)]$$

$$= (p+q)F_{n+1} + pF_n - (p+q)$$

$$= (pF_{n+2} + qF_{n+1}) - (p+q)$$

$$\sum_{r=0}^{n} h_r(p,q) = h_{n+2} - (p+q) \text{ by [2]} \qquad \cdots (1_{iii})$$

1, 1_i, (1_{ii}), (1_{iii}) can be proved for all (+ve) integers n by induction. We shall here prove (1) as an illustration. Let us suppose that

$$(1)' \qquad \sum_{r=0}^{k} w_r = a + \frac{bT_k - aqT_{k-1}}{1 - p + q}$$

Next let us add w_{k+1} to both sides, to get

$$\sum_{r=0}^{k+1} w_r = a + \frac{bT_k - aqT_{k-1}}{1 - p + q} + w_{k+1}$$

$$= a + \frac{b(1 - u_k + qu_{k-1}) - aq(1 - u_{k-1} + qu_{k-2})}{1 - p + q}$$

$$+ A\alpha^{k+1} + B\beta^{k+1}$$

$$(4) \qquad \sum_{r=0}^{k+1} w_r = a + \frac{b(1 - u_k + qu_{k-1}) - aq(1 - u_{k-1} + qu_{k-2})}{1 - p + q}$$

$$+ bu_k - aqu_{k-1}$$

$$= a + \frac{1}{1 - p + q}[b(1 - u_{k+1} + qu_k) - aq(1 - u_k + qu_{k-1})]$$

$$\sum_{r=0}^{k+1} w_r = a + \frac{bT_{k+1} - aqT_k}{1 - p + q}$$

Equation (4) is of the same form as (1)' with k replaced by k + 1. Hence, etc.

Similarly other results can be proved for all positive integral values of n.

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