Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contribution are asked to enclose self-addressed stamped postcards.

B-160 Proposed by Robert H. Anglin, Dan River Mills, Danville, Virginia.

Show that, if \( x = F_n F_{n+3} \), \( y = 2F_{n+1} F_{n+2} \), and \( z = F_{2n+3} \), then \( x^2 + y^2 = z^2 \).

B-161 Proposed by John Ivie, Student at University of California, Berkeley, California

Given the Pell numbers defined by \( P_{n+2} = 2P_{n+1} + P_n \), \( P_0 = 0 \), \( P_1 = 1 \), show that for \( k > 0 \);

\[
P_k = \sum_{r=0}^{\left(\frac{k-1}{2}\right)} \left( \frac{k}{2r + 1} \right) 2^r.
\]

\[
P_{2k} = \sum_{r=1}^{k} \left( \frac{k}{r} \right) 2^r P_r.
\]

B-162 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California

Let \( r \) be a fixed positive integer and let the sequences \( u_1, u_2, \cdots \) satisfy \( u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-r} \) for \( n > r \) and have initial conditions \( u_j = 2^{j-1} \) for \( j = 1, 2, \cdots, r \). Show that every representation of \( U_n \) as a sum
of distinct u_j must be of the form u_n itself or contain explicitly the terms 
\( u_{n-1}, u_{n-2}, \ldots, u_{n-r+1} \) and some representation of \( u_{n-r} \).

B-163 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico

Let \( n \) be a positive integer. Clearly

\[
(1 + \sqrt{5})^n = a_n + b_n \sqrt{5},
\]

with \( a_n \) and \( b_n \) integers. Show that \( 2^{n-1} \) is a divisor of \( a_n \) and of \( b_n \).

B-164 Proposed by J. A. H. Hunter, Toronto, Canada.

A Fibonacci-type sequence is defined by:

\[
G_{n+2} = G_{n+1} + G_n,
\]

with \( G_1 = a \) and \( G_2 = b \). Find the minimum positive values of integers \( a \) and \( b \), subject to \( a \) being odd, to satisfy:

\[
G_{n-1} G_{n+1} - G_n^2 = -11111(-1)^n \quad \text{for } n > 1.
\]

B-165 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Define the sequence \( \{b(n)\} \) by

\[
b(1) = b(2) = 1, \quad b(2k) = b(k), \quad \text{and} \quad b(2k + 1) = b(k + 1) + b(k)
\]

for \( k \geq 1 \). For \( n \geq 1 \), show the following:

(a) \( b\left(\left[2^{n+1} + (-1)^n\right]/3\right) = F_{n+1}. \)

(b) \( b\left(\left[7 \cdot 2^{n-1} + (-1)^n\right]/3\right) = L_n. \)
SOLUTIONS

A MULTIPLICATIVE ANALOGUE

B-142 Proposed by William D. Jackson, SUNY at Buffalo, Amherst, N.Y.

Define a sequence as follows: \( A_1 = 2, \ A_2 = 3, \) and \( A_n = A_{n-1}A_{n-2} \)
for \( n > 2 \). Find an expression for \( A_n \).

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

Let \( B_n = \ln A_n \) for \( n \geq 1 \). Then \( B_1 = \ln 2, \ B_2 = \ln 3 \) and \( B_n = B_{n-1} + B_{n-2} \) for \( n > 2 \). Clearly

\[
B_n = F_{n-2} \cdot \ln 2 + F_{n-1} \cdot \ln 3
\]

for \( n > 2 \), or

\[
A_n = 2^{F_{n-2}} \cdot 3^{F_{n-1}}
\]

for \( n > 2 \).


THE DETERMINANT VANISHES

B-143 Proposed by Raphael Finkelstein, Tempe, Arizona.

Show that the following determinant vanishes when \( a \) and \( d \) are natural numbers:

\[
\begin{vmatrix}
F_a & F_{a+d} & F_{a+2d} \\
F_{a+3d} & F_{a+4d} & F_{a+5d} \\
F_{a+6d} & F_{a+7d} & F_{a+8d}
\end{vmatrix}
\]

What is the value of the determinant one obtains by replacing each Fibonacci number by the corresponding Lucas number?
Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico

Let \( r = \frac{F_{6d}}{F_{3d}} \) and \( s = F_{6d+1} - rF_{3d+1} \). Then

\[
\begin{align*}
rf_{3d} + sF_0 &= F_{6d} \\
tF_{3d+1} + sF_1 &= F_{6d+1}
\end{align*}
\]

It follows by induction that

\[
F_{n+6d} = rf_{n+3d} + sF_n
\]

for all \( n \); in particular, it is true for \( n = a, \ n = a + d, \) and \( n = a + 2d \).

Hence the three rows of the matrix are linearly dependent and the determinant is zero.

If each Fibonacci number is replaced by the corresponding Lucas number, the determinant will also be zero by similar reasoning.

\textbf{Editorial Note:} It can be shown that \( r = \frac{L_{3d}}{F_{3d}} \) and \( s = (-1)^{d+1} \).

\textit{Also solved by F. D. Parker, C. B. A. Peck, David Zeitlin and the proposer.}

\textbf{LUCAS ALPHAMETIC}

B-144 Proposed by J. A•H. Hunter, Toronto, Canada.

In this alphametic each distinct letter stands for a particular but different digit, all ten digits being represented here. It must be the Lucas series, but what is the value of the SERIES?

\[
\begin{align*}
\text{ONE} \\
\text{THREE} \\
\text{START} \\
\text{L} \\
\text{SERIES}
\end{align*}
\]

\textit{Solution by Charles W. Trigg, San Diego, California}

Since they are the initial digits of integers, none of \( \theta, T, S, \) or \( L \) can be zero. Proceeding from the left, clearly \( S = 1, \ E = 0, \) and \( T \) is 8 or 9. In either event, \( H + T > 10, \) so \( T = 8. \) Then from the units' column, \( L = 3. \)
The three integer columns then establish the equalities:

\[
\begin{align*}
N + R + 1 &= 10 \\
\theta + R + A + 1 &= I + 10 \\
H + 8 + 1 &= R + 10 
\end{align*}
\]

Whereupon, \(N + H = 10\) and \((N,H) = (4,6)\) or \((6,4)\). But \(H = R + 1\), so \(R = 5\), \(H = 6\), \(N = 4\).

Then \(\theta + A = I + 4\), and \(\theta = 9\), \(A = 2\), \(I = 7\). (\(\theta\) and \(A\) may be interchanged.) Consequently, SERIES = 105701.


**BINARY N-TUPLES**

B-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Given an unlimited supply of each of two distinct types of objects, let \(f(n)\) be the number of permutations of \(n\) of these objects such that no three consecutive objectives are alike. Show that \(f(n) = 2F_{n+1}\), where \(F_n\) is the \(n\)th Fibonacci number.

Solution by Bruce W. King, Adirondack Community College, Glen Falls, N.Y.

Call a permutation of the required type an 'admissible \(n\) permutation,' and let \(A\) and \(B\) be two of the distinct types of objects. A list of admissible \(n + 1\) permutations can be constructed in the following way:

(a) For each admissible \(n\) permutation ending in \(A\), adjoin \(B\) on the right; for each distinct admissible \(n\) permutation ending in \(B\), adjoin on the right.

(b) For each distinct admissible \(n - 1\) permutation ending in \(A\), adjoin \(BB\) on the right; for each distinct admissible \(n - 1\) permutation ending in \(B\), adjoin \(AA\) on the right.

Certainly the resulting list contains only admissible \(n + 1\) permutations. Furthermore, there is no possibility of duplication because the permutations described in (b) end with two identical letters, but those described in (a) end with two different letters. Lastly, no \(n + 1\) permutation is unobtainable in
this way. For, if there were such a permutation, either the \( n - 1 \) permutation excluding its last two letters, or the \( n \) permutation excluding its last letter would have to be admissible. Consequently, we see that

\[ f(n + 1) = f(n - 1) + f(n). \]

The rest is an easy proof by induction.

By direct enumeration, \( f(3) = 6 = 2F_4 \). If \( f(n) = 2F_{n+1} \) for integers \( n \leq N \), then

\[ f(N + 1) = f(N - 1) + f(N) = 2F_N + 2F_{N+1} = 2(F_N + F_{N+1}) = 2F_{N+2}, \]

and the proof is complete.

Also solved by J. L. Brown, Jr., C. B. A. Peck, Michael Yoder, and the proposer.

ANGLES OF A TRIANGLE

B-146 Proposed by Walter W. Horner, Pittsburgh, Pennsylvania

Show that \( \pi = \arctan \left( \frac{1}{F_{2n}} \right) + \arctan \frac{1}{F_{2n+1}} + \arctan \frac{1}{F_{2n+2}} \).


From the solution to H-82 (FQ, 6, 1, 52-54), we get

\[ \arctan \left( \frac{1}{F_{2n}} \right) = \arctan \left( \frac{1}{F_{2n+2}} \right) + \arctan \left( \frac{1}{F_{2n+1}} \right). \]

The result now follows from \( \arctan x + \arctan \left( \frac{1}{x} \right) = \pi/2 \).

Also solved by Herta T. Freitag, John Ivie, Bruce W. King, John Wessner, Gregory Wulczyn, Michael Yoder, and the proposer.

TWIN PRIMES


Let
be the sum of the sum of the reciprocals of all twin primes below $2^{15}$. Indicate which of the following inequalities is true:

(a) $S < \pi^2/6$  
(b) $\pi^2/6 < S < \sqrt{e}$  
(c) $\sqrt{e} < S$.

Solutions by Paul Sands, Student, University of New Mexico, Albuquerque, New Mexico, and the proposer. (Both used electronic computers.)

<table>
<thead>
<tr>
<th>True inequality</th>
<th>Proposer True number</th>
<th>Sands True number</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b)</td>
<td>55</td>
<td>55</td>
</tr>
<tr>
<td>S, to six decimal places</td>
<td>1.647986</td>
<td>1.648627</td>
</tr>
</tbody>
</table>

(Continued from p. 210.)

6. $T_n = (-1)^R$

7. $T_n+1 = 5T_n - 6T_{n-1}$
   $T_n = 2^n + 3^{n-1}$

8. $r = \frac{5 + \sqrt{29}}{2}$, $s = \frac{5 - \sqrt{29}}{2}$
   $T_n = r^n - s^n$ with terms 1, 5, 26, 135, · · ·
   $V_n = r^n + s^n$ with terms 5, 27, 140, · · ·

9. $r = \frac{3 + i\sqrt{11}}{2}$, $s = \frac{3 - i\sqrt{11}}{2}$
   $T_n = \left(\frac{33 - 16i\sqrt{11}}{55}\right)r^n + \left(\frac{33 + 16i\sqrt{11}}{55}\right)s^n$

10. $T_{n+1} = 5T_n + 2T_{n-1}$; $T_1 = 3$, $T_2 = 7$.  

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