SOME RESULTS ON FIBONACCI QUATERNIONS

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1. INTRODUCTION

Recently the author derived some results about generalized Fibonacci Numbers [3]. In the present paper our object is to derive relations connecting the Fibonacci Quaternions [1] and Lucas Quaternions, to use a similar terminology, with the Fibonacci Numbers [2] and Lucas Numbers [4] as also the inter-relations between them. In Section 3, we give relations connecting Fibonacci and Lucas Numbers; in Section 4, we derive relations of Fibonacci Quaternions to Fibonacci and Lucas Numbers, and in 5, Lucas Quaternions are connected to Fibonacci and Lucas Numbers. Lastly, in Section 6 are listed the relations existing between Fibonacci and Lucas Quaternions.

2. TERMINOLOGY AND NOTATIONS

Following the terminology of A. F. Horadam [1], we define the n^{th} Fibonacci Quaternion as follows:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

where F_n is the n^{th} Fibonacci Number and i, j, k satisfy the relations of the Quaternion viz:

$$i^2 = j^2 = k^2 = -1, \quad ij - ji = k; \quad jk = -kj = i; \quad ki = -ik = j.$$

Now on the same lines we can define the n^{th} Lucas Quaternion T_n say as

$$T_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}$$

where L_n is the n^{th} Lucas number. Also, we will denote a quantity of the form

$$F_n - iF_{n-1} + jF_{n-2} - kF_{n-3}$$

by Q_{n^*} and

$$F_n + iF_{n-1} + jF_{n-2} + kF_{n-3}$$

by $Q_{\bar{n}}$. Similar notations hold for T_{n^*} and $T_{\bar{n}}$, that is,

$$L_n - iL_{n-1} + jL_{n-2} - kL_{n-3} = T_{n^*}$$

and

$$L_n + iL_{n-1} + jL_{n-2} + kL_{n-3} = T_{\bar{n}} .$$

Now we proceed to derive the relations one by one. All these results are obtained from the definitions of Fibonacci Numbers and Lucas Numbers, given by

$$F_n = \frac{a^n - b^n}{\sqrt{5}} , \quad L_n = (a^n + b^n)$$

for all n, where a and b are the roots of the equation

$$x^2 - x - 1 = 0 ,$$

obtained from the Fibonacci and Lucas recurrence relations. The roots are connected by

$$a + b = 1, \quad a - b = \sqrt{5} ,$$

and $ab = -1$.

SECTION 3

Consider the following relations:

(1) $$F_{n+r}L_{n+r} = F_{2n+2r}$$

(2) $$F_{n-r}L_{n-r} = F_{2n-2r}$$

Therefore

(3) $$F_{n+r}L_{n+r} + F_{n-r}L_{n-r} = F_{2n}L_{2r}$$

(4) $$F_{n+r}L_{n+r} - F_{n-r}L_{n-r} = F_{2r}L_{2n}$$

(5) $$F_{n+r}L_{n-r} = F_{2n} + (-1)^{n-r}F_{2r}$$

(6) $$F_{n-r}L_{n+r} = F_{2n} - (-1)^{n-r}F_{2r}$$

Therefore

(7) $$F_{n+r}L_{n-r} + F_{n-r}L_{n+r} = 2F_{2n}$$

and

(8) $$F_{n+r}L_{n-r} - F_{n-r}L_{n+r} = 2(-1)^{n-r}F_{2r}$$

(9) $$F_{n+r}L_{n} = F_{2n+r} + (-1)^{n}F_{r}$$

(10) $$F_{n}L_{n+r} = F_{2n+r} - (-1)^{n}F_{r}$$

So

(11) $$F_{n+r}L_{n} + F_{n}L_{n+r} = 2F_{2n+r}$$

(12) $$F_{n+r}L_{n} - F_{n}L_{n+r} = 2(-1)^{n}F_{r}$$

(13) $$F_{n+r}L_{n+s} = F_{2n+r+s} + (-1)^{n+s}F_{r-s}$$

(14) $$F_{n+s}L_{n+r} = F_{2n+r+s} + (-1)^{n+s+1}F_{r-s}$$

(15) $$F_{n+r}L_{n+s} + F_{n+s}L_{n+r} = 2F_{2n+r+s}$$

(16) $$F_{n+r}L_{n+s} - F_{n+s}L_{n+r} = 2(-1)^{n+s}F_{r-s}$$

SECTION 4

In this section, we give the list of relations connecting the Fibonacci Quaternions to Fibonacci and Lucas Numbers. The simplest one is

(17) $$Q_n - iQ_{n+1} - jQ_{n+2} - kQ_{n+3} = L_{n+3}$$

Consider

(18) $$Q_{n-1}Q_{n+1} - Q_n^2 = (-1)^n\left[2Q_1 - 3k\right]$$

(19) $$Q_{n-1}^2 + Q_n^2 = 2Q_{2n-1} - 3L_{2n+2}$$

(20) $$Q_{n+1}^2 - Q_{n-1}^2 = Q_nT_n = (2Q_{2n} - 3L_{2n+3}) + 2(-1)^{n+1}(Q_0 - 3k)$$

(21) $$Q_{n-2}Q_{n-1} + Q_nQ_{n+1} = 6F_nQ_{n-1} - 9F_{2n+2} + 2(-1)^{n+1}(Q_{(-1)} - 3k)$$

(22) $$Q_{n-1}Q_{n+3} - Q_{n+1}^2 = (-1)^n\left[2 + 4i + 3j + k\right]$$

(23) $$Q_{n-1}Q_{n+1} - Q_{n-2}Q_{n+2} = (-1)^n\left[2T_0 - k\right] + 4(-1)^{n+1}\left[Q_0 - 2k\right]$$

(24) $$Q_{n-3}Q_{n-2} + Q_nQ_{n+1} = 4Q_{2n-2} - 6L_{2n+1}$$

(25) $$Q_{n-1}^2 + Q_{n+1}^2 = 6F_{n+1}Q_{n-1} - 9F_{2n+3} + 2(-1)^nQ_{(-2)}$$

Also the remarkable relation

(26) $$\frac{Q_{n+r} + (-1)^rQ_{n-r}}{Q_n} = L_r$$

(27) $$Q_{n+1-r}Q_{n+1+r} - Q_{n+1}^2 = (-1)^{n-r}[F_r^2T_0 + F_{2r}(Q_0 - 3r)]$$

Now we turn to relations of the form:

(28) $$Q_{n+r}L_{n+r} = Q_{2n+2r} + (-1)^{n+r}Q_0$$

(29) $$Q_{n-r}L_{n-r} = Q_{2n-2r} + (-1)^{n+r}Q_0$$

(30) $$Q_{n+r}L_{n+r} + Q_{n-r}L_{n-r} = Q_{2n}L_{2r} + 2(-1)^{n+r}Q_0$$

(31) $$Q_{n+r}L_{n+r} - Q_{n-r}L_{n-r} = F_{2r}T_{2n}$$

(32) $$Q_{n+r}L_{n-r} = Q_{2n} + (-1)^{n-r}Q_{2r}$$

(33) $$Q_{n-r}L_{n+r} = Q_{2n} + (-1)^{n-r+1}Q_{2r^*}$$

(34) $$Q_{n+r}L_{n-r} + Q_{n-r}L_{n+r} = 2Q_{2n} + (-1)^{n-r}L_{2r}Q_0$$

(35) $$Q_{n+r}L_{n-r} - Q_{n-r}L_{n+r} = (-1)^{n-r}F_{2r}T_0$$

(36) $$Q_{n+r}L_n = Q_{2n+r} + (-1)^nQ_r$$

(37) $$Q_nL_{n+r} = Q_{2n+r} - (-1)^nQ_r^*$$

(38) $$Q_{n+r}L_n + Q_nL_{n+r} = 2Q_{2n+r} + (-1)^nL_rQ_0$$

(39) $$Q_{n+r}L_n - Q_nL_{n+r} = (-1)^nF_rT_0$$

(40) $$Q_{n+r}L_{n+t} = Q_{2n+r+t} + (-1)^{n+t}Q_{r-t}$$

(41) $$Q_{n+t}L_{n+r} = Q_{2n+r+t} + (-1)^{n+r+1}Q_{\overline{r-t}}$$

Therefore:

(42) $$Q_{n+r}L_{n+t} + Q_{n+t}L_{n+r} = 2Q_{2n+r+t} + (-1)^{n+t}L_{r-t}Q_0$$

(43) $$Q_{n+r}L_{n+t} - Q_{n+t}L_{n+r} = (-1)^{n+t}F_{r-t}T_0$$

(44) $$Q_{n+r}F_{n-r} = \frac{1}{5}\left[T_{2n} - (-1)^{n-r}T_{2r}\right]$$

(45) $$Q_{n-r}F_{n+r} = \frac{1}{5}\left[T_{2n} - (-1)^{n-r}T^{\star}_{2r}\right]$$

(46) $$Q_{n+r}F_{n-r} + Q_{n-r}F_{n+r} = \frac{1}{5}\left[2T_{2n} - (-1)^{n-r}L_{2r}T_0\right]$$

(47) $$Q_{n+r}F_{n-r} - Q_{n-r}F_{n+r} = (-1)^{n-r+1}F_{2r}Q_0$$

(48) $$Q_{n+r}F_n = \frac{1}{5}\left[T_{2n+r} - (-1)^nT_r\right]$$

(49) $$Q_nF_{n+r} = \frac{1}{5}\left[T_{2n+r} - (-1)^nT^{\star}_r\right]$$

(50) $$Q_{n+r}F_n + Q_nF_{n+r} = \frac{1}{5}\left[2T_{2n+r} - (-1)^nL_rT_0\right]$$

(51) $$Q_{n+r}F_n - Q_nF_{n+r} = (-1)^{n+1}F_rQ_0$$

(52) $$Q_{n+r}F_{n+t} = \frac{1}{5}\left[T_{2n+r+t} - (-1)^{n+t}T_{r-t}\right]$$

(53) $$Q_{n+t}F_{n+r} = \frac{1}{5}\left[T_{2n+r+t} - (-1)^{n+r+1}T_{\overline{r-t}}\right]$$

(54) $$Q_{n+r}F_{n+t} + Q_{n+t}F_{n+r} = \frac{1}{5}\left[2T_{2n+r+t} - (-1)^{n+t}L_{r-t}T_0\right]$$

(55) $$Q_{n+r}F_{n+t} - Q_{n+t}F_{n+r} = (-1)^{n+t}F_{r-t}Q_0$$

(56) $$Q_{n+r}F_{n-r} + (-1)^rQ_{n-r}F_{n+r} = \frac{1}{5}\left[T_{2n}(1+(-1)^r) - (-1)^{n-r}T_{2r} - (-1)^nT^{\star}_{2r}\right]$$

(57) $$Q_{n+r}L_{n-r} + (-1)^rQ_{n-r}L_{n+r} = Q_{2n}(1+(-1)^r) + (-1)^{n-r}Q_{2r} - (-1)^nQ^{\star}_{2r}$$

(58) $$Q_{n+r}L_{n+t} + (-1)^rQ_{n+t}L_{n+r} = Q_{2n+r+t}(1+(-1)^r) + (-1)^{n+t}Q_{r-t} - (-1)^{n+r+t}Q^{\star}_{r-t}$$

(59) $$Q_{n+r}F_{n+t} + (-1)^r Q_{n+t}F_{n+r} = \frac{1}{5}\Big[T_{2n+r+t}(1 + (-1)^r) - (-1)^{n+t}T_{r-t} - (-1)^{n+r+t}T^{*}_{r-t}\Big]$$

SECTION 5

In this section we give the results connecting Lucas Quaternions T_n to Fibonacci and Lucas Numbers. The simplest is:

(60) $$T_n - iT_{n+1} - jT_{n+2} - kT_{n+3} = 15F_{n+3}$$

(61) $$T_{n+r}F_{n+r} = Q_{2n+2r} - (-1)^{n+r}Q_0$$

(62) $$T_{n-r}F_{n-r} = Q_{2n-2r} - (-1)^{n+r}Q_0$$

(63) $$T_{n+r}F_{n+r} + T_{n-r}F_{n-r} = Q_{2n}L_{2r} - 2(-1)^{n+r}Q_0$$

(64) $$T_{n+r}F_{n+r} - T_{n-r}F_{n-r} = F_{2r}T_{2n}$$

(65) $$T_{n+r}F_{n-r} = Q_{2n} - (-1)^{n-r}Q_{2r}$$

(66) $$T_{n-r}F_{n+r} = Q_{2n} + (-1)^{n-r}Q^{*}_{2r}$$

(67) $$T_{n+r}F_{n-r} + T_{n-r}F_{n+r} = 2Q_{2n} - (-1)^{n-r}L_{2r}Q_0$$

(68) $$T_{n+r}F_{n-r} - T_{n-r}F_{n+r} = (-1)^{n+r+1}F_{2r}T_0$$

(69) $$T_{n+r}F_n = Q_{2n+r} - (-1)^n Q_r$$

(70) $$T_nF_{n+r} = Q_{2n+r} + (-1)^n Q^{*}_r$$

(71) $$T_{n+r}F_n + T_nF_{n+r} = 2Q_{2n+r} - (-1)^n L_r Q_0$$

(72) $$T_{n+r}F_n - T_nF_{n+r} = (-1)^{n+1}F_rT_0$$

(73) $$T_{n+r}F_{n+t} = Q_{2n+r+t} - (-1)^{n+t}Q_{r-t}$$

(74) $$T_{r+t}F_{n+r} = Q_{2n+r+t} + (-1)^{n+r}Q_{\overline{r-t}}$$

So

(75) $$T_{n+r}F_{n+t} + T_{n+t}F_{n+r} = 2Q_{2n+r+t} - (-1)^{n+t}L_{r-t}Q_0$$

(76) $$T_{n+r}F_{n+t} - T_{n+t}F_{n+r} = (-1)^{n+t+1}F_{r-t}T_0$$

(77) $$T_{n+r}L_{n-r} = T_{2n} + (-1)^{n-r}T_{2r}$$

(78) $$T_{n-r}L_{n+r} = T_{2n} + (-1)^{n-r}T_{2r}^{\star}$$

(79) $$T_{n+r}L_{n-r} + T_{n-r}L_{n+r} = 2T_{2n} + (-1)^{n-r}L_{2r}T_0$$

(80) $$T_{n+r}L_{n-r} - T_{n-r}L_{n+r} = (-1)^{n-r}5F_{2r}Q_0$$

(81) $$T_{n+r}L_n = T_{2n+r} + (-1)^n T_r$$

(82) $$T_n L_{n+r} = T_{2n+r} + (-1)^n T_r^{\star}$$

(83) $$T_{n+r}L_n + T_n L_{n+r} = 2T_{2n+r} + (-1)^n L_r T_0$$

(84) $$T_{n+r}L_n - T_n L_{n+r} = (-1)^n 5F_r Q_0$$

(85) $$T_{n+r}L_{n+t} = T_{2n+r+t} + (-1)^{n+t}T_{r-t}$$

(86) $$T_{n+t}L_{n+r} = T_{2n+r+t} + (-1)^{n+r+1}T_{\overline{r-t}}$$

(87) $$T_{n+r}L_{n+t} + T_{n+t}L_{n+r} = 2T_{2n+r+t} + (-1)^{n+t}L_{r-t}T_0$$

(88) $$T_{n+r}L_{n+t} - T_{n+t}L_{n+r} = (-1)^{n+t+1}5F_{r-t}Q_0$$

(89) $$T_{n+r}L_{n-r} + (-1)^r T_{n-r}L_{n+r} = T_{2n}(1 + (-1)^r) + (-1)^{n-r}T_{2r} - (-1)^n T_{2r}^{\star}$$

(90) $$T_{n+r}F_{n-r} + (-1)^r T_{n-r}F_{n+r} = \frac{1}{5}\Big[Q_{2n}(1 + (-1)^r) - (-1)^{n-r}Q_{2r} + (-1)^n Q^\star_{2r}\Big]$$

(91) $$T_{n+r}F_{n+t} + (-1)^r T_{n+t}F_{n+r} = \frac{1}{5}\Big[Q_{2n+r+t}(1 + (-1)^r) - (-1)^{n+t}Q_{r-t} + (-1)^{n+r+t}Q^\star_{r-t}\Big]$$

(92) $$T_{n+r}L_{n+t} + (-1)^r T_{n+t}L_{n+r} = T_{2n+r+t}(1 + (-1)^r) + (-1)^{n+t}T_{r-t} + (-1)^{n+r+t}T^\star_{r-t}$$

SECTION 6

Lastly, in this section we obtain the inter-relations between the Fibonacci and Lucas Quaternions

(93) $$Q_nL_n + T_nF_n = 2Q_{2n}$$

(94) $$Q_nL_n - T_nF_n = 2(-1)^n Q_0$$

(95) $$Q_n + T_n = 2Q_{n+1}$$

(96) $$T_n - Q_n = 2Q_{n-1}$$

(97) $$T_n^2 + Q_n^2 = 6\Big[2F_nQ_n - 3F_{2n+3}\Big] + 4(-1)^n T_0$$

(98) $$T_n^2 - Q_n^2 = 4\Big[2F_nQ_n - 3F_{2n+3} + (-1)^n T_0\Big]$$

(99) $$T_nQ_n + T_{n-1}Q_{n-1} = 2T_{2n-1} - 15F_{2n+2}$$

(100) $$T_nQ_n - T_{n-1}Q_{n-1} = 2Q_{2n-1} - 3L_{2n+2} + 4(-1)^n(Q_0 - 3k)$$

(101) $$T_nQ_{n+1} - T_{n+1}Q_n = 2(-1)^n\Big[2\,Q_1 - 3k\Big]$$

(102) $$T_{n+r}Q_{n+s} - T_{n+s}Q_{n+r} = 2(-1)^{n+s+1}F_{r-s}T_0$$

REFERENCES

1. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quaternions," Amer. Math. Monthly, 70, 1963, pp. 289-291.
2. A. F. Horadam, "A Generalized Fibonacci Sequence," Amer. Math. Monthly, 68, 1961, pp. 455-459.
3. Muthulakshmi R. Iyer, "Identities Involving Generalized Fibonacci Numbers," the Fibonacci Quarterly, Vol. 7, No. 1 (Feb. 1969), pp. 66-72.
4. E. Lucas, Theorie des Numbers, Paris, 1961.

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(Continued from p. 200.)

SOLUTIONS TO PROBLEMS

1. For any modulus m, there are m possible residues $(0, 1, 2, \cdots, m-1)$. Successive pairs may come in m^2 ways. Two successive residues determine all residues thereafter. Now in an infinite sequence of residues there is bound to be repetition and hence periodicity.

Since m divides T_0, it must by reason of periodicity divide an infinity of members of the sequence.

2. $n = mk$, where m and k are odd. V_n can be written

$$V_n = (r^m)^k + (s^m)^k \quad ,$$

which is divisible by $V_m = r^m + s^m$.

3. $r = 2 + 2i\sqrt{2}, \quad s = 2 - 2i\sqrt{2}$.

$$T_n = \left(\frac{2 - 3i\sqrt{2}}{16}\right) r^n + \left(\frac{2 + 3i\sqrt{2}}{16}\right) s^n \quad .$$

4. The auxiliary equation is $(x-1)^2 = 0$, so that T_n has the form

$$T_n = An \times 1^n + B \times 1^n = An + B \,.$$

5. $$T_n = 2^n \left[\left(\frac{b - 2a}{4}\right) n + \frac{4a - b}{4}\right] \quad .$$

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