LINEAR RECURSION RELATIONS — LESSON FIVE
RECURSION RELATIONS OF HIGHER ORDER

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The considerations applied to linear recursion relations of the second order form a pattern for dealing with relations of higher order. Given a linear recursion relation of the $k^{th}$ order:

\[ T_{n+1} = a_1 T_n + a_2 T_{n-1} + \cdots + a_k T_{n-k+1} , \]

where the quantities $a_i$ and $T_i$ are real, there would be an auxiliary equation

\[ x^k - a_1 x^{k-1} - a_2 x^{k-2} \cdots - a_k = 0 , \]

for which there could be real and distinct roots, multiple real roots or complex roots conjugate in pairs. The major difficulty that arises in a relation of this type is the problem of determining the roots which ordinarily would be approximate in value.

As an example, consider one extension of the Fibonacci sequences, namely, adding the last three terms, or adding the last four terms, and so on. The recursion relations and corresponding auxiliary equations would be:

\[ T_{n+1} = T_n + T_{n-1} + T_{n-2} \quad \text{and} \quad x^3 - x^2 - 1 = 0 \]  

\[ T_{n+1} = T_n + T_{n-1} + T_{n-2} + T_{n-3} \quad \text{and} \quad x^4 - x^3 - x^2 - x - 1 = 0 . \]

If we look at the general type of this equation:

\[ x^k = x^{k-1} + x^{k-2} + x^{k-3} + \cdots + x + 1 , \]

it appears that since

\[ 2^k - 1 = 2^{k-1} + 2^{k-2} + 2^{k-3} + \cdots + 2 + 1 , \]
there should be a root near 2. The following table gives an approximation to
this root for various values of \( k \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>Approximation to Root near 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1.83928676</td>
</tr>
<tr>
<td>4</td>
<td>1.92756198</td>
</tr>
<tr>
<td>5</td>
<td>1.96594824</td>
</tr>
<tr>
<td>6</td>
<td>1.98358285</td>
</tr>
<tr>
<td>7</td>
<td>1.99196420</td>
</tr>
<tr>
<td>8</td>
<td>1.99603118</td>
</tr>
<tr>
<td>9</td>
<td>1.99802948</td>
</tr>
</tbody>
</table>

Approximations, such as these, to real or complex roots can be determined,
but expressing \( T_n \) in terms of them does not seem very satisfying. Never-
theless, as will be seen in a subsequent lesson, such evaluations of roots of
the auxiliary equation provide interesting information regarding the generated
sequence.

**MULTIPLE ROOTS**

The case of multiple roots calls for additional consideration. If a poly-
nomial equation

\[
(7) \quad a_0 x^k + a_1 x^{k-1} + a_2 x^{k-2} + \ldots + a_k = 0
\]

has a root of multiplicity \( s \), then (7) can be written:

\[
(8) \quad (x - r)^s F(x) = 0 ,
\]

where \( F(x) \) is a polynomial of degree \( k - s \). Clearly, this equation and the
equations formed by setting the first \( s - 1 \) derivatives equal to zero are all
satisfied by \( r \). This provides a clue for dealing with roots of any multiplicity
when found in the auxiliary equation of a recursion relation. For concreteness,
let us consider a root \( r \) of multiplicity 3.
Let the equation having this multiple root be:

\[
(9) \quad x^3 - ax^2 - bx - c = 0 .
\]

Multiply both sides of the equation by \(x^n\) to obtain:

\[
(10) \quad x^{n+3} - ax^{n+2} - bx^{n+1} - cx^n = 0 .
\]

Take the derivative and set the resulting polynomial equal to zero.

\[
(11) \quad (n + 3)x^{n+2} - a(n + 2)x^{n+1} - b(n + 1)x^n - cnx^{n-1} = 0 .
\]

Repeat this operation on (11).

\[
(12) \quad (n + 3)(n + 2)x^{n+1} - a(n + 2)(n + 1)x^n - b(n + 1)nx^{n-1} - cnx^{n-2} = 0 .
\]

The multiple root \(r\) must satisfy the relations (10), (11), and (12) so that on replacing \(x\) by \(r\) and multiplying (11) by \(r\) and (12) by \(r^2\) we have the following three recursion relations for \(r\).

\[
(13) \quad r^{n+3} = ar^{n+2} + br^{n+1} + cr^n ,
\]

\[
(14) \quad (n + 3)r^{n+3} = a(n + 2)r^{n+2} + b(n + 1)r^{n+1} + cnr^n ,
\]

\[
(15) \quad (n + 3)(n + 2)r^{n+3} = a(n + 2)(n + 1)r^{n+2} + b(n + 1)nr^{n+1} + cn(n - 1)r^n .
\]

On the basis of these recursion relations the indicated expression for \(T_n\) is:

\[
(16) \quad T_n = An(n - 1)r^n + Bnr^n + Cr^n .
\]

We show first that this relation continues to hold for succeeding values of \(n\) if it is true for three consecutive values. For if

\[
(17) \quad T_{n+1} = A(n + 1)n r^{n+1} + B(n + 1)r^{n+1} + Cr^{n+1} ,
\]
and

\[ T_{n+2} = A(n + 2)(n + 1)r^{n+2} + B(n + 2)r^{n+2} + C r^{n+2}, \]

then

\[ T_{n+3} = a T_{n+2} + b T_{n+1} + c T_{n} \]

is equal to:

\[ T_{n+3} = A(n + 3)(n + 2)r^{n+3} + B(n + 3)r^{n+3} + C r^{n+3} \]

on the basis of relations (13), (14), and (15).

Given three initial values \( T_1, T_2, \) and \( T_3 \), the relations they should satisfy on the basis of (16) would be:

\[
\begin{align*}
T_1 &= Br + Cr \\
T_2 &= 2Ar^2 + 2Br^2 + Cr^2 \\
T_3 &= 6Ar^3 + 3Br^3 + Cr^3 
\end{align*}
\]

The determinant of the coefficients of the unknowns \( A, B, C \) has a value of \(-2r^2\), so that if \( r \) is not zero, there are unique solutions for \( A, B, \) and \( C \). Thus three initial values \( T_1, T_2, \) and \( T_3 \) can be expressed in the form given by (16). It follows that this form will continue to hold for all values of \( n \).

It may be noted in passing that if the multiple root has a value of \( 1 \), \( T_n \) reduces to a polynomial in \( n \).

**Example.** Express the terms of the recursion relation

\[ T_{n+1} = 7T_n - 17T_{n-1} + 14T_{n-2} + 4T_{n-3} - 8T_{n-4} \]

in terms of the roots of the auxiliary equation:

\[ x^5 - 7x^4 + 17x^3 - 14x^2 - 4x + 8 = 0. \]
By synthetic division three equal roots, 2, are found and the residual quadratic has the roots

\[ \frac{1 + \sqrt{5}}{2} \text{ and } \frac{1 - \sqrt{5}}{2}. \]

Accordingly,

\[ T_{n+1} = A(n - 1)x^n + Bnx^n + Cx^n + Dr^n + Es^n \]

where

\[ r = \frac{1 + \sqrt{5}}{2} \text{ and } s = \frac{1 - \sqrt{5}}{2} \]

**Example.** For the recursion relation

\[ T_{n+1} = 3T_n - 3T_{n-1} + T_{n-2}, \]

with initial values \( T_1 = 5, T_2 = 8, T_3 = 17 \), express \( T_n \) in terms of the roots of the auxiliary equation.

This equation is

\[ x^3 - 3x^2 + 3x - 1 = 0, \]

which has a triple root of 1. Thus

\[ T_{n+1} = An(n - 1) + Bn + C, \]

a polynomial in \( n \). Then

\[
\begin{align*}
5 &= B + C \\
8 &= 2A + 2B + C \\
17 &= 6A + 3B + C,
\end{align*}
\]
leading to the values \( A = 3, \ B = -3, \ C = 8 \), so that

\[
T_{n+1} = 3n^2 - 6n + 8.
\]

**PROBLEMS**

1. Find the recursion relation satisfied by

\[
T_n = 3n^2 - 5n + 4 + 2 \cdot 5^n.
\]

2. Given the recursion relation

\[
T_{n+1} = 6T_n - 11T_{n-1} + 6T_{n-2},
\]

and initial values

\[
T_1 = 8, \quad T_2 = 15, \quad T_3 = 22.
\]

Express the general term \( T_n \) in terms of the roots of the auxiliary equation.

3. \( S_n \) is the Fibonacci sequence 3, 7, 10, 17, 27, \( \cdots \), and \( R_n \) is the geometric progression 5, 15, 45, 135, \( \cdots \)

\[
T_n = R_n + S_n.
\]

Find the recursion relation for \( T_n \).

4. If \( T_n = 3n + 2 + 2(-1)^n + F_n \), find the recursion relation for \( T_n \).

5. If \( T_1 = 13, \ T_2 = 15, \ T_3 = 22 \) and \( T_{n+1} = 4T_n - T_{n-1} - 2T_{n-2} \) express \( T_n \) in terms of the roots of the auxiliary equation of this recursion relation.

(Solutions are on p. 302.)

**NOTICE**

The two fine elementary books, *The Introduction to Fibonacci Discovery* and *Fibonacci and Lucas Numbers*, are each available for $1.50 from Brother Alfred Brousseau, St. Mary's College, California 94575.