Suggested by David Zeitlin’s solutions to B-148, 149, and 150.

Let a and b be distinct numbers, \( U_n = (a^n - b^n)/(a - b) \), and \( V_n = a^n + b^n \). Establish generalizations of the formulas

(a) \[ F_{(2^n)} = F_n L_n L_{2n} \cdots L_{(2-1)n} \]

(b) \[ L_{n+1} L_{n+3} + 4(-1)^{n+1} = 5F_n F_{n+1} \]

of B-148 and B-149 in which one deals with \( U_n \) and \( V_n \) instead of \( F_n \) and \( L_n \).

Proposed by A. G. Shannon, University of Papua and New Guinea, Boroko, T. P. N. G.

Let \( L_n \) be the \( n \)th Lucas number defined by \( L_1 = 1 \), \( L_2 = 3 \), and \( L_{n+2} = L_{n+1} + L_n \) for \( n \geq 1 \). For which values of \( n \) is

\[ nL_{n+1} > (n + 1)L_n \]?

Proposed by S. H. L. King, Jacksonville University, Jacksonville, Florida.

Using each of six of the nine positive digits 1, 2, *, 3, 4, 5, 6, 7, and 8 exactly once, form an integer \( z \) such that each of \( z, 2z, 3z, 4z, 5z, \) and \( 6z \) contains the same six digits once and only once.
Prove the following identities:

(a) \[ F_n^4 + F_{n-1}^4 + F_{n+1}^4 = 2(F_n^4 F_{n-1}^4 - F_{n+1}^2)^2 \]

(b) \[ F_n^5 + F_{n-1}^5 - F_{n+1}^5 = 5F_{n-1}^5 F_{n+1}(F_n^5 F_{n-1}^5 - F_{n+1}^2) \]

where \( F_1 = F_2 = 1 \) and \( F_{n+1} = F_n + F_{n-1} \). Show that these are two cases of an infinite sequence of identities.

Let the binomial coefficient \( \binom{m}{r} \) be zero when \( m < r \) and let

\[ S_n = \sum_{j=0}^{\infty} (-1)^j \binom{n-j}{j} . \]

Show that \( S_{n+2} - S_{n+1} + S_n = 0 \) and hence \( S_{n+3} = -S_n \) for \( n = 0, 1, 2, \ldots \).

Let \( \binom{m}{r} = 0 \) for \( m < r \) and let

\[ T_n = \sum_{j=0}^{\infty} \binom{n-2j}{2j} . \]

Obtain a fourth-order homogeneous linear recurrence formula for \( T_n \).

Solutions

Correction. In the solution to B-128 in Vol. 6, No. 4 (Oct. 1968), line 2 from the bottom of p. 295 should read:

\[ S_{4n} = f_{4n+2} - f_2 = (F_{4n+1}^4 - 1)f_2 + F_{4n}^f_1 , \]
and line 5 from the top of p. 296 should read:

$$F_{4n+1} - 1 = F_{2n} L_{2n+1}.$$ 

COMMENT. Mr. J. D. E. Konhauser, Macalester College, St. Paul, Minnesota, sent in the following on B-130a:


ADDITIONS TO LISTS OF SOLVERS: Problem B-143 was also solved by D. V. Jaiswal (Indore, India), Amanda Neel, and A. G. Shannon (Boroko, T. P. N. G.) Problem B-143 was also solved by D. V. Jaiswal and A. G. Shannon. Problem B-146 was also solved by D. V. Jaiswal and A. G. Shannon.

TELESCOPING PRODUCT


Let $F_n$ and $L_n$ denote the Fibonacci and Lucas numbers and show that

$$F_{(2n)}^t = F_n L_n L_{2n} L_{4n} \cdots L_{(2^t - 1)n}.$$ 


By the well-known formula $F_{2n} = F_n L_n$, we have

$$F_{(2n)}^t = F_{2t-1} L_{2t-1} = F_{2t-2} L_{2t-2} \cdots = F_{n} L_{n} L_{2n} \cdots L_{2t-1}.$$ 

Also solved by Christine Anderson, Serge Hamelin (Canada), Bruce W. King, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, Michael Yoder, David Zeitlin, and the proposer.

A QUADRATIC IDENTITY

B-149 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that
\[ L_{n+1} L_{n+3} + 4(-1)^{n+1} = 5F_n F_{n+4}. \]

Solution by Carol A. Vespe, Student, University of New Mexico, Albuquerque, New Mexico.

Let \( a = \frac{1 + \sqrt{5}}{2} \) and \( b = \frac{1 - \sqrt{5}}{2}. \) Since both sides of the equation are of the form

\[ c_1(a^2)^n + c_2(ab)^n + c_3(b^2)^n, \]

with constant \( c_1, \) it suffices to note that the identity holds for \( n = 0, 1, \) and 2.

Also solved by Clyde A. Bridger, Juliette Davenport, Herta T. Freitag, Serge Hamelin (Canada), Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind, John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), C. C. Yalavigi (Mercara, India), Michael Yoder, David Zeitlin, and the Proposer.

**ANOTHER QUADRATIC IDENTITY**

B-150 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Show that

\[ L_n^2 - F_n^2 = 4F_{n-1}F_{n+1}. \]

Solution by David Zeitlin, Minneapolis, Minnesota.

Let \( U_n \) and \( V_n \) be solutions of \( W_{n+2} = aW_{n+1} + bW_n, \) where \( U_0 = 0, \) \( U_1 = 1, \) \( V_0 = 2, \) and \( V_1 = a. \) Noting that

\[ V_n = 2U_{n+1} - aU_n \equiv U_{n+1} + bU_{n-1}. \]

we obtain

(1) \[ V_n^2 - a^2 U_n^2 = 4bU_{n-1}U_{n+1}. \]

The desired result is obtained from (1) with \( a = b = 1, \) \( V_n \equiv L_n, \) and \( U_n \equiv F_n. \)
Also solved by Clyde A. Bridger, Juliette Davenport, David Englund, Herta T. Freitag, Serge Hamelin (Canada), John E. Homer, Jr., Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind (England), John W. Milsom, C. B. A. Peck, Gerald Satlow, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, Michael Yoder, and the Proposer.

MISSING TERMS

B-151 Proposed by Hal Leonard, San Jose State College, San Jose, California.

Let \( m = L_1 + L_2 + \cdots + L_n \) be the sum of the first \( n \) Lucas numbers.

Let

\[
P_n(x) = \prod_{i=1}^{n} (1 + x^i) = a_0 + a_1x + \cdots + a_m x^m.
\]

Let \( q_n \) be the number of integers \( k \) such that both \( 0 < k < m \) and \( a_k = 0 \).

Find a recurrence relation for the \( q_n \).

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Note that

\[
m = m_n = L_1 + L_2 + \cdots + L_n = L_{n+2} - 3
\]

and that \( q_n \) is the number of integers in \( \{1, 2, 3, \cdots, m - 1\} \) that are not expressible in the form

\[
c_1 L_1 + c_2 L_2 + \cdots + c_n L_n ; \quad c_i \in \{0, 1\} \quad \text{for} \quad 1 \leq i \leq n.
\]


Using

\[
m_{2k} = L_1 + L_2 + L_3 + \cdots + L_{2k} = L_2 + L_5 + L_7 + \cdots + L_{2k+1}
\]
\[
m_{2k-1} = L_1 + L_2 + \cdots + L_{2k-1} = L_1 + (L_4 + L_6 + \cdots + L_{2k})
\]
and formula (43) on page 303 of Klarner's paper, one has

\[ q_{2k} = m_{2k} - (F_4 + F_6 + \cdots + F_{2k+2}) = L_{2k+2} - 3 - (F_{2k+3} - F_3) \]
\[ q_{2k-1} = m_{2k-1} - F_2 - (F_6 + F_7 + \cdots + F_{2k+1}) = L_{2k+1} - 3 - 1 - (F_{2k+2} - F_4) \]

Now \( L_n = F_{n+1} + F_{n-1} \) leads to \( q_n = F_{n+1} - 1 \) for all \( n \). Hence

\[ (q_{n+2} + 1) = (q_{n+1} + 1) + (q_n + 1) \quad \text{or} \quad q_{n+2} = q_{n+1} + q_n + 1 \]

Also solved by Serge Hamelin (Quebec, Canada), C. B. A. Peck, and Carol A. Vespe. Hamelin gave the homogeneous recursion formula \( q_{n+3} = 2q_{n+2} - q_n \).

**FIBONACCI ADDITION FORMULA**

B-152 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Prove that

\[ F_{m+n} = F_{m+1}F_{n+1} - F_{m-1}F_{n-1} \]

Solution by John E. Homer, Jr., Lisle, Illinois.

From the well-known formulas

\[ F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n \]
\[ F_{m+n-1} = F_mF_n + F_{m-1}F_{n-1} \]

we have

\[ F_{m+1}F_{n+1} - F_{m-1}F_{n-1} = F_{m+n+1} - F_{m+n-1} = F_{m+n} \]

Also solved by Clyde A. Bridger, Juliette Davenport, David Englund, Herta T. Freitag, Serge Hamelin (Canada), Bruce W. King, H. V. Krishna (Manipal, India), Douglas Lind (England), John W. Milsom, C. B. A. Peck, A. G. Shannon (Boroko, T. P. N. G.), Carol A. Vespe, C. C. Yalavigi (Merrara, India), Michael Yoder, and the Proposer.

[Continued in p. 276.]

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