A NOTE ON FIBONACCI QUATERNIONS

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A. F. Horadam has derived in [1] some results regarding Fibonacci and generalized Fibonacci quaternions. The object of this note is to derive some more relations connecting these two quaternions. Following [1] Q_n and P_n are defined as

$$\text{(1a)} \qquad Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}$$

$$\text{(1b)} \qquad P_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}$$

where

$$\text{(1c)} \qquad i^2 = j^2 = k^2 = -1, \quad ij = -jk = k, \quad jk = -kj = i, \quad ki = -ik = j\,.$$

Let us now consider the relation

$$P_n + qQ_n = [H_n + iH_{n+1} + jH_{n+2} + kH_{n+3}] + q\,[F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}].$$

Also from (1) of [1] we have

$$H_n = (p - q)F_n + qF_{n+1}\,,$$

so

$$\begin{aligned} P_n + qQ_n = {} & [(p-q)F_n + qF_{n+1}] + i\,[(p-q)F_{n+1} + qF_{n+2}] \\ & + j\,[(p-q)F_{n+2} + qF_{n+3}] + k\,[(p-q)F_{n+3} + qF_{n+4}] \\ & + q\,[F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}]. \end{aligned}$$

This becomes after some simplifications

$$\begin{aligned} = {} & p(F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}) \\ & + q(F_{n+1} + iF_{n+2} + jF_{n+3} + kF_{n+4}) \end{aligned}$$

Hence,

$$P_n + qQ_n = pQ_n + qQ_{n+1}$$

or

$$(2) \qquad \begin{aligned} P_n &= pQ_n + q(Q_{n+1} - Q_n) \\ P_n &= pQ_n + qQ_{n-1} \end{aligned}$$

by definition of Q_n. Consider now the quantity

$$P_n\overline{Q}_n - \overline{P}_nQ_n \quad ,$$

where $\overline{P}_n$, $\overline{Q}_n$ are conjugate quaternions respectively of P_n and Q_n.

$$\begin{aligned} P_n\overline{Q}_n - \overline{P}_nQ_n &= (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3})(F_n - iF_{n+1} - jF_{n+2} - kF_{n+3}) \\ &\quad - (H_n - iH_{n+1} - jH_{n+2} - kH_{n+3})(F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}) \\ &= -2H_n(Q_n - F_n) + 2F_n(P_n - H_n) \end{aligned}$$

$$(3a) \qquad P_n\overline{Q}_n - \overline{P}_nQ_n = 2(F_nP_n - H_nQ_n)$$

Dividing by $P_nQ_n \neq 0$,

$$(3b) \qquad \begin{aligned} \frac{\overline{Q}_n}{Q_n} - \frac{\overline{P}_n}{P_n} &= 2\left(\frac{F_n}{Q_n} - \frac{H_n}{P_n}\right) \\ \frac{\overline{Q}_n - 2F_n}{Q_n} &= \frac{\overline{P}_n - 2H_n}{P_n} \end{aligned}$$

Again,

$$\begin{aligned} P_n\overline{Q}_n + \overline{P}_nQ_n &= 2\sum_{i=0}^{3} H_{n+i}F_{n+i} - 2iH_{n+1}(jF_{n+2} + kF_{n+3}) \\ &\quad - 2jH_{n+2}(iF_{n+1} + kF_{n+3}) - 2kH_{n+3}(iF_{n+1} + jF_{n+2}) \end{aligned}$$

Using (1c) and simplifying we have,

$$= 2H_nF_n + 2H_{n+1}(F_{n+1} - kF_{n+2} + jF_{n+3})$$
$$+ 2H_{n+2}(F_{n+2} + kF_{n+1} - iF_{n+3}) + 2H_{n+3}(F_{n+3} - jF_{n+1} + iF_{n+2})$$

Now using

$$i^2 = j^2 = k^2 = -1 ,$$

we may write the above relation as

$$P_n\overline{Q}_n + \overline{P}_nQ_n = 2H_nF_n - 2[iH_{n+1} + jH_{n+2} + kH_{n+3}][iF_{n+1} + jF_{n+2} + kF_{n+3}]$$
$$= 2H_nF_n - 2(P_n - H_n)(Q_n - F_n)$$

$$P_n\overline{Q}_n + \overline{P}_nQ_n = -2(P_nQ_n - P_nF_n - Q_nH_n)$$

(4) $$P_n\overline{Q}_n + \overline{P}_nQ_n = 2[P_nF_n + Q_nH_n - P_nQ_n]$$

As P_n and $Q_n \neq 0$, dividing by P_nQ_n,

$$\frac{\overline{Q}_n}{Q_n} + \frac{\overline{P}_n}{P_n} = 2\left[\frac{F_n}{Q_n} + \frac{H_n}{P_n} - 1\right]$$

$$\frac{\overline{Q}_n - 2F_n}{Q_n} + 1 = \frac{2H_n - \overline{P}_n}{P_n} - 1$$

or

(4b) $$\frac{\overline{Q}_n - 2F_n}{Q_n} + 1 = -\left[\frac{\overline{P}_n - 2H_n}{P_n} + 1\right]$$

Also

$$P_nQ_n - \overline{P}_n\overline{Q}_n = (H_n + iH_{n+1} + jH_{n+2} + kH_{n+3})(F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})$$
$$- (H_n - iH_{n+1} - jH_{n+2} - kH_{n+3})(F_n - iF_{n+1} - jF_{n+2} - kF_{n+3})$$
$$= 2F_n(iH_{n+1} + jH_{n+2} + kH_{n+3})$$
$$+ 2H_n(iF_{n+1} + jF_{n+2} + kF_{n+3})$$
$$= 2F_n(P_n - H_n) + 2H_n(Q_n - F_n)$$

(4)
$$P_nQ_n - \overline{P}_n\overline{Q}_n = 2[H_nQ_n + F_nP_n - 2H_nF_n]$$

Theorem:

$$Q_{n-1}^2 + Q_n^2 = 2Q_{2n-1} - 3L_{2n+2}$$

Let us consider the left side of the relation.

$$Q_{n-1}^2 + Q_n^2 = (F_{n-1} + iF_n + jF_{n+1} + kF_{n+2})^2 + (F_n + iF_{n+1} + jF_{n+2} + kF_{n+3})^2$$
$$= [F_{n-1}^2 - F_n^2 - F_{n+1}^2 - F_{n+2}^2 + F_n^2 - F_{n+1}^2 - F_{n+2}^2 - F_{n+3}^2]$$
$$+2[F_{n-1}(iF_n + jF_{n+1} + kF_{n+2}) + F_n(iF_{n+1} + jF_{n+2} + kF_{n+3})]$$
$$+ [iF_{n+1}(jF_{n+2} + kF_{n+3}) + jF_{n+2}(iF_{n+1} + kF_{n+3})]$$
$$+ [iF_n(jF_{n+1} + kF_{n+2}) + jF_{n+1}(iF_n + kF_{n+2})]$$
$$+ [kF_{n+2}(iF_n + jF_{n+1}) + kF_{n+3}(iF_{n+1} + jF_{n+2})]$$

The first term

$$= F_{n-1}^2 - F_{n+1}^2 - (F_{n+1}^2 + F_{n+2}^2) - (F_{n+2}^2 + F_{n+3}^2)$$

(A)
$$= -[(F_{n+1}^2 - F_{n-1}^2) + (F_{n+1}^2 + F_{n+2}^2) + (F_{n+2}^2 + F_{n+3}^2)]$$
$$= -[(F_{2n} + F_{2n+3} + F_{2n+5}]$$
$$= -[12F_{2n} + 7F_{2n-1}]$$

Now consider the terms containing i, j, k, namely,

$$\text{(B)}\quad \begin{aligned} &2i\left[F_nF_{n+1} + F_nF_{n+1}\right] + 2j\left[F_{n-1}F_{n+1} + F_nF_{n+2}\right] \\ &\qquad + 2k\left[F_{n-1}F_{n+2} + F_nF_{n+3}\right] \\ &= 2iF_{2n} + 2jF_{2n+1} + 2kF_{2n+2} \end{aligned}$$

Therefore

$$\begin{aligned} Q^2_{n-1} + Q^2_n &= -[12F_{2n} + 7F_{2n-1}] \\ &\quad + 2iF_{2n} + 2jF_{2n+1} + 2kF_{2n+2} \\ &= -[12F_{2n} + 9F_{2n-1} - 2F_{2n-1}] \\ &\quad + 2iF_{2n} + jF_{2n+1} + kF_{2n+2} \\ &= -[3F_{2n+3} + 3F_{2n+1}] + 2Q_{2n-1} \\ &= -3L_{2n+2} + 2Q_{2n=1} \end{aligned}$$

Hence,

$$Q^2_{n-1} + Q^2_n = 2Q_{2n-1} - 3L_{2n+2}$$

Hence the theorem.

Other interesting relations will be considered later.

REFERENCES

1. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quaternions," Amer. Math. Monthly, 70, 3, 1963.

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