ON DETERMINANTS INVOLVING GENERALIZED FIBONACCI NUMBERS

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In this note we shall evaluate some determinants whose elements are the Generalized Fibonacci numbers, T_n, defined by the relations:

$$T_1 = a, \qquad T_2 = b, \qquad T_{n+2} = T_{n+1} + T_n .$$

We can express

$$T_n = C\alpha^n + D\beta^n,$$

where α,β are the roots of the equation $X^2 - X - 1 = 0$, and C and D are constants. The Fibonacci numbers, F_n, are obtained by taking $a = b = 1$, and the Lucas numbers, L_n, by taking $a = 1$, $b = 3$.

We shall make use of the following well known identities:

(i) $$F_{-n} = (-1)^{n-1}F_n ,$$

(ii) $$T_{m+n} = T_mF_{n+1} + T_{m-1}F_n ,$$

(iii) $$T^2_{n+1} - T^2_{n-1} = aT_{2n-2} + bT_{2n-1} ,$$

(iv) $$T_{m-1}T_n - T_mT_{n-1} = (-1)^{m-1}F_{n-m}D ,$$

and shall also use the formulae,

(v) $$T_{m+r}F_{n+r} + (-1)^{r+1}T_mF_n = T_{m+n+r}F_r ,$$

The truth of this formulae can be established, either by induction over r, or by substituting the values of F_n and T_n in terms of α and β.

1. THIRD-ORDER DETERMINANT

We shall show that

$$(1.1) \qquad \begin{vmatrix} T_p & T_{p+m} & T_{p+m+n} \\ T_q & T_{q+m} & T_{q+m+n} \\ T_r & T_{r+m} & T_{r+m+n} \end{vmatrix} = 0 \quad ,$$

for all integers p, q, r, m, and n. Using (ii), we can write

$$T_{k+m+n} = T_{k+m}F_{n+1} + T_{k+m-1}F_n, \quad (k = p,q,r)$$

hence the determinant on the left-hand side can be written as

$$F_{n+1} \begin{vmatrix} T_p & T_{p+m} & T_{p+m} \\ T_q & T_{q+m} & T_{q+m} \\ T_r & T_{r+m} & T_{r+m} \end{vmatrix} + F_n \begin{vmatrix} T_p & T_{p+m} & T_{p+m-1} \\ T_q & T_{q+m} & T_{q+m-1} \\ T_r & T_{r+m} & T_{r+m-1} \end{vmatrix}$$

Obviously the first determinant vanishes. The second, on subtracting the elements of the 3rd column from those of the 2nd, reduces to

$$F_n \begin{vmatrix} T_p & T_{p+m-2} & T_{p+m-1} \\ T_q & T_{q+m-2} & T_{q+m-1} \\ T_r & T_{r+m-2} & T_{r+m-1} \end{vmatrix} .$$

Now on subtracting the elements of the 2nd column from the 3rd, we obtain

$$F_n \begin{vmatrix} T_p & T_{p+m-2} & T_{p+m-3} \\ T_q & T_{q+m-2} & T_{q+m-3} \\ T_r & T_{r+m-2} & T_{r+m-3} \end{vmatrix} .$$

Thus alternately subtracting the 2nd and the 3rd columns from one another, the process can be continued to reduce the suffixes. At a certain stage, if m is even, 1st and 2nd columns will become identical; and if m is odd, 1st and 3rd columns will become identical. Hence for every value of m, even or odd, the determinant vanishes.

2. EVALUATION OF THE DETERMINANT

We shall now evaluate the determinant,

$$\Delta \equiv \begin{vmatrix} T_p + k & T_{p+m} + k & T_{p+m+n} + k \\ T_q + k & T_{q+m} + k & T_{q+m+n} + k \\ T_r + k & T_{r+m} + k & T_{r+m+n} + k \end{vmatrix} ,$$

where k is an arbitrary constant, and p, q, r, m, and n are integers.

On writing the determinant as the sum of eight determinants, and using (1.1) and the property that a determinant vanishes if two columns are identical, we obtain

$$\Delta \equiv \begin{vmatrix} T_p & T_{p+m} & k \\ T_q & T_{q+m} & k \\ T_r & T_{r+m} & k \end{vmatrix} + \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix} + \begin{vmatrix} \cdots \\ \cdots \\ \cdots \end{vmatrix} ,$$

$$= K \cdot F_m \begin{vmatrix} T_p & T_{p-1} & 1 \\ T_q & T_{q-1} & 1 \\ T_r & T_{r-1} & 1 \end{vmatrix} + \cdots + \cdots .$$

The first determinant by using (iv) can be written as

$$= D \cdot K \cdot F_m \left[(-1)^{r-1} F_{q-r} + (-1)^{p-1} F_{r-p} + (-1)^{q-1} F_{p-q} \right] .$$

Hence

$$\Delta = D \cdot K \left[(-1)^q F_{r-q} - (-1)^p F_{r-p} + (-1)^p F_{q-p} \right] \times \times \left[F_m - F_{m+n} + (-1)^m F_n \right] . \tag{2.1}$$

3. FOURTH-ORDER DETERMINANTS

We shall now evaluate the determinant,

$$\Delta \equiv \begin{vmatrix} T_{n+3} & T_{n+2} & T_{n+1} & T_n \\ T_{n+2} & T_{n+3} & T_n & T_{n+1} \\ T_{n+1} & T_n & T_{n+3} & T_{n+2} \\ T_n & T_{n+1} & T_{n+2} & T_{n+3} \end{vmatrix}$$

It can be easily shown that the determinant,

$$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = \left[(a + b)^2 - (c + d)^2\right] \cdot \left[(a - b)^2 - (c - d)^2\right] .$$

Hence we obtain

$$\begin{aligned} \Delta &= \left[(T_{n+3} + T_{n+2})^2 - (T_{n+1} + T_n)^2\right] \times \\ &\qquad \times \left[(T_{n+3} - T_{n+2})^2 - (T_{n+1} - T_n)^2\right] \\ &= (T^2_{n+4} - T^2_{n+2}) \cdot (T^2_{n+1} - T^2_{n-1}) \\ &= (aT_{2n+4} + bT_{2n+5}) \cdot (aT_{2n-2} + bT_{2n-1}) \quad , \end{aligned} \tag{3.1}$$

on using (iii).

4. EVALUATING THE CIRCULANT

We now evaluate the circulant,

$$\begin{vmatrix} T_n & T_{n+k} & \cdots & T_{n+(m-1)k} \\ T_{n+(m-1)k} & T_n & \cdots & T_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ T_{n+k} & T_{n+2k} & \cdots & T_n \end{vmatrix} .$$

Let w be any one of the m numbers

$$w_r = \cos \frac{2r\pi}{m} + i \sin \frac{2r\pi}{m} , \quad (r = 1, 2, 3, \cdots, m)$$

so that $w^m = 1$, and

$$\begin{aligned} S_1 &\equiv w_1 + w_2 + w_3 + \cdots + w_m = 0 \\ S_2 &\equiv w_1 w_2 + \quad \cdots \quad = 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ S_m &\equiv w_1 w_2 w_3 w_4 \quad \cdots \quad w_m = (-1) \cdot (-1)^m = (-1)^{m+1} . \end{aligned}$$

Hence we get

$$\prod_{r=1}^{m} (y - w_r z) = y^m - z^m . \tag{4.1}$$

Therefore as discussed in [8],

$$\Delta = \prod_{r=1}^{m} (T_n + w_r T_{n+k} + \cdots + w_r^{m-1} T_{n+(m-1)k})$$

$$= \prod_{r=1}^{m} \left[\frac{C\alpha^n(1 - w_r^m \alpha^{mk})}{1 - w_r \alpha^k} + \frac{D\beta^n(1 - w_r^m \beta^{mk})}{1 - w_r \beta^k} \right]$$

$$= \prod_{r=1}^{m} \left[\frac{(T_n - T_{n+mk}) - (-1)^k w_r (T_{n-k} - T_{n+(m-1)k})}{(1 - w_r \alpha^k)(1 - w_r \beta^k)} \right] \tag{4.2}$$

$$= \frac{(T_n - T_{n+mk})^m - (-1)^{mk}(T_{n-k} - T_{n+(m-1)k})^m}{(1 - \alpha^{mk})(1 - \beta^{mk})}$$

$$= \frac{(T_n - T_{n+mk})^m - (-1)^{mk}(T_{n-k} - T_{n+(m-1)k})^m}{1 + (-1)^{mk} - L_{mk}}$$

5. EACH ELEMENT IS THE PRODUCT OF TWO NUMBERS

We shall evaluate

$$\Delta \equiv \begin{vmatrix} F_n \cdot T_{m+n} & F_{n+p} \cdot T_{m+n+p} & F_{n+p+q} \cdot T_{m+n+p+q} \\ F_{n+r} \cdot T_{m+n+r} & F_{n+r+p} \cdot T_{m+n+r+p} & F_{n+r+p+q} \cdot T_{m+n+r+p+q} \\ F_{n+s} \cdot T_{m+n+s} & F_{n+s+p} \cdot T_{m+n+s+p} & F_{n+s+p+q} \cdot T_{m+n+s+p+q} \end{vmatrix},$$

and shall show that $|\Delta|$ is independent of n.

On using (v), we can write

$$F_{n+p}T_{m+n+p} + (-1)^{p+1}F_nT_{m+n} = F_pT_{m+2n+p} .$$

Hence multiplying 1st column by $(-1)^{p+1}$, $(-1)^{p+q+1}$, and adding respectively to the 2nd and 3rd columns, we obtain

$$\Delta = F_pF_{p+q}\begin{vmatrix} F_n \cdot T_{m+n} & T_{m+2n+p} & T_{m+2n+p+q} \\ F_{n+r} \cdot T_{m+n+r} & T_{m+2n+2r+p} & T_{m+2n+2r+p+q} \\ F_{n+s} \cdot T_{m+n+s} & T_{m+2n+2s+p} & T_{m+2n+2s+p+q} \end{vmatrix}$$

$$= F_pF_{p+q}F_q\begin{vmatrix} F_n T_{m+n} & T_{m+2n+p} & T_{m+2n+p-1} \\ F_{n+r}T_{m+n+r} & T_{m+2n+2r+p} & T_{m+2n+2r+p-1} \\ F_{n+s}T_{m+n+s} & T_{m+2n+2s+p} & T_{m+2n+2s+p-1} \end{vmatrix}$$

on using (ii).

Now alternately subtracting the 3rd and 2nd columns from one another, we can write

$$\Delta = F_pF_qF_{p+q}(-1)^{m+p}\begin{vmatrix} F_n \cdot T_{m+n} & T_0 & T_1 \\ F_{n+r} \cdot T_{m+n+r} & T_{2r} & T_{2r+1} \\ F_{n+s} \cdot T_{m+n+s} & T_{2s} & T_{2s+1} \end{vmatrix}$$

$$= F_pF_qF_{p+q}(-1)^{m+p} \cdot D\Big[F_nT_{m+n}F_{2s-2r} - F_{n+r}T_{m+n+r}F_{2s} + F_{n+s}T_{m+n+s}F_{2r}\Big]$$

on using (iv).

Now on expressing the numbers in terms of α and β, we can write

$$F_{n+s}T_{m+n+s}F_{2r} = \frac{1}{5}\Big[T_{m+2n+2s+2r} - T_{m+2n+2s-2r} + (-1)^{n+s}(T_{m-2r} - T_{m+2r})\Big]$$

Hence we have

$$\text{(5.1)} \qquad \Delta = \frac{1}{5}F_pF_qF_{p+q}(-1)^{m+n+p} \cdot D\Big[(T_{m+2r-2s} - T_{m+2s-2r}) + (-1)^s(T_{m-2r} - T_{m+2r}) - (-1)^r(T_{m-2s} - T_{m+2s})\Big]$$

Also it is obvious that $|\Delta|$ is independent of n.

6. ONCE AGAIN THE FOURTH ORDER

We shall now show that

$$(6.1)\qquad \Delta \equiv \begin{vmatrix} F_pT_{p+m} & F_{p+a}T_{p+m+a} & F_{p+b}T_{p+m+b} & F_{p+c}T_{p+m+c} \\ F_qT_{q+m} & F_{q+a}T_{q+m+a} & F_{q+b}T_{q+m+b} & F_{q+c}T_{q+m+c} \\ F_rT_{r+m} & F_{r+a}T_{r+m+a} & F_{r+b}T_{r+m+b} & F_{r+c}T_{r+m+c} \\ F_sT_{s+m} & F_{s+a}T_{s+m+a} & F_{s+b}T_{s+m+b} & F_{s+c}T_{s+m+c} \end{vmatrix} = 0,$$

for all integers p, q, r, s, m, a, b, and c.

Multiplying 1st column by $(-1)^{a+1}$, $(-1)^{b+1}$, $(-1)^{c+1}$ and adding to the 2nd, 3rd, and 4th columns, respectively; and using the formula (v), the determinant reduces to

$$F_a \cdot F_b \cdot F_c \cdot \begin{vmatrix} F_pT_{p+m} & T_{2p+m+a} & T_{2p+m+b} & T_{2p+m+c} \\ F_qT_{q+m} & T_{2q+m+a} & T_{2q+m+b} & T_{2q+m+c} \\ F_rT_{r+m} & T_{2r+m+a} & T_{2r+m+b} & T_{2r+m+c} \\ F_sT_{s+m} & T_{2s+m+a} & T_{2s+m+b} & T_{2s+m+c} \end{vmatrix} .$$

Expanding along the 1st column and using the result (1.1), the determinant vanishes. This can be generalized for the n^{th} order determinants.

7. PARTICULAR CASES

A. Let us take a = b = 1, then $T_n \equiv F_n$ and D = -1.

(i) On putting m = n in (1.1), we get

$$\begin{vmatrix} F_p & F_{p+n} & F_{p+2n} \\ F_q & F_{q+n} & F_{q+2n} \\ F_r & F_{r+n} & F_{r+2n} \end{vmatrix} = 0$$

— a problem suggested by Vladimir Ivanoff [4].

(ii) On taking p = a, q = a + 3d, r = a + 6d, m = n = d in (1.1), we get

$$\begin{vmatrix} F_a & F_{a+d} & F_{a+2d} \\ F_{a+3d} & F_{a+4d} & F_{a+5d} \\ F_{a+6d} & F_{a+7d} & F_{a+8d} \end{vmatrix} = 0$$

— a problem suggested by Raphael Finkelstein [7].

(iii) On taking $p = n$, $q = n + 1$, $r = n + 2$, $m = n = 1$ in (2.1), we get

(7.1)
$$\begin{vmatrix} F_n + k & F_{n+1} + k & F_{n+2} + k \\ F_{n+1} + k & F_{n+2} + k & F_{n+3} + k \\ F_{n+2} + k & F_{n+3} + k & F_{n+4} + k \end{vmatrix}$$
$$= (-1)\cdot k\cdot [(-1)^{n+1} - (-1)^n + (-1)^n]\times$$
$$\times [F_1 - F_1 - F_2]$$
$$= k\cdot(-1)^{n+1}$$

— a problem suggested by Brother U. Alfred [2].

(iv) We obtain from (3.1)

$$\begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{vmatrix} = F_{2n+6}\cdot F_{2n}$$

— a problem suggested by George Ledin [5].

(v) We obtain from (4.1)

$$\begin{vmatrix} F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\ F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ F_{n+k} & F_{n+2k} & \cdots & F_n \end{vmatrix}$$
$$= \frac{(F_n - F_{n+mk})^m - (-1)^{mk}(F_{n-k} - F_{n+(m-1)k})^m}{1 - L_{mk} + (-1)^{mk}}$$

— a problem suggested by L. Carlitz [6].

(vi) On taking $m = 0$ in (5.1), we get

$$\begin{vmatrix} F_n^2 & F_{n+p}^2 & F_{n+p+q}^2 \\ F_{n+r}^2 & F_{n+r+p}^2 & F_{n+r+p+q}^2 \\ F_{n+s}^2 & F_{n+s+p}^2 & F_{n+s+p+q}^2 \end{vmatrix}$$

$$= \frac{2}{5} \cdot F_p \cdot F_q \cdot F_{p+q} (-1)^{n+p} [F_{2s-2r} + (-1)^s F_{2r} - (-1)^r F_{2s}]$$

on using result (i).

(vi)-(a) On substituting $p = q = 1$, $r = 1$, $s = 2$, we get

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix}$$

$$= \frac{2}{5} (-1)^{n+1} (F_2 + F_2 + F_4)$$

$$= 2(-1)^{n+1}$$

— a problem suggested by Brother U. Alfred [1].

(vi)-(b) On substituting $p = q = 2$, $r = 2$, $s = 4$, we get

$$\begin{vmatrix} F_n^2 & F_{n+2}^2 & F_{n+4}^2 \\ F_{n+2}^2 & F_{n+4}^2 & F_{n+6}^2 \\ F_{n+4}^2 & F_{n+6}^2 & F_{n+8}^2 \end{vmatrix}$$

$$= \frac{2}{5} \cdot 3 \cdot (-1)^n \cdot (3 + 3 - 21)$$

$$= 18 (-1)^{n+1}$$

— a problem suggested by Brother U. Alfred [3].

(vii) On taking $m = 1$ in (5.1), we obtain

$$\begin{vmatrix} F_nF_{n+1} & F_{n+p}F_{n+p+1} & F_{n+p+q}F_{n+p+q+1} \\ F_{n+r}F_{n+r+1} & F_{n+r+p}F_{n+r+p+1} & F_{n+r+p+q}F_{n+r+p+q+1} \\ F_{n+s}F_{n+s+1} & F_{n+s+p}F_{n+s+p+1} & F_{n+s+p+q}F_{n+s+p+q+1} \end{vmatrix}$$

$$= \frac{1}{5}\, F_r \cdot F_q \cdot F_{p+q}(-1)^{n+p}\,[(F_{2r-2s+1} - F_{1+2s-2r}) + $$

$$+ (-1)^s(F_{1-2r} - F_{1+2r}) - (-1)^r(F_{1-2s} - F_{1+2s})]$$

$$= \frac{1}{5}\, F_pF_qF_{p+q}\,(-1)^{n+p}[-(F_{2s-2r+1} - F_{2s-2r-1}) + $$

$$+ (-1)^{s+1}(F_{2r+1} - F_{2r-1}) + (-1)^r(F_{2s+1} - F_{2s-1})]$$

$$= \frac{1}{5}\,(-1)^{n+p+1}\, F_pF_qF_{p+q}[F_{2s-2r} + (-1)^sF_{2r} - (-1)^r\, F_{2s}]\,.$$

(vii)-(a) On taking $p = q = r = 1$, and $s = 2$, we have

$$\begin{vmatrix} F_nF_{n+1} & F_{n+1}F_{n+2} & F_{n+2}F_{n+3} \\ F_{n+1}F_{n+2} & F_{n+2}F_{n+3} & F_{n+3}F_{n+4} \\ F_{n+2}F_{n+3} & F_{n+3}F_{n+4} & F_{n+4}F_{n+5} \end{vmatrix}$$

$$= \frac{1}{5}\,(-1)^n\,(F_2 + F_2 + F_4)$$

$$= (-1)^n\,.$$

(viii) On taking $m = 0$ in (6.1), we get

$$\begin{vmatrix} F_p^2 & F_{p+a}^2 & F_{p+b}^2 & F_{p+c}^2 \\ F_q^2 & F_{q+a}^2 & F_{q+b}^2 & F_{q+c}^2 \\ F_r^2 & F_{r+a}^2 & F_{r+b}^2 & F_{r+c}^2 \\ F_s^2 & F_{s+a}^2 & F_{s+b}^2 & F_{s+c}^2 \end{vmatrix} = 0\,,$$

for all integers p, q, r, s, a, b, and c.

B. On taking $a = 1$, $b = 3$, we have $T_n \equiv L_n$ and $D = 5$.

(i) On taking $p = a$, $q = a + 3d$, $r = a + 6d$, $m = n = d$ in (1.1), we get

$$\begin{vmatrix} L_a & L_{a+d} & L_{a+2d} \\ L_{a+3d} & L_{a+4d} & L_{a+5d} \\ L_{a+6d} & L_{a+7d} & L_{a+8d} \end{vmatrix} = 0$$

— a problem suggested by Raphael Finkelstein [7].

(ii) We obtain from (2.1) that

$$\begin{vmatrix} L_p + k & L_{p+m} + k & L_{p+n} + k \\ L_q + k & L_{q+m} + k & L_{q+n} + k \\ L_r + k & L_{r+m} + k & L_{r+n} + k \end{vmatrix}$$

$$= -5 \cdot \begin{vmatrix} F_p + k & F_{p+m} + k & F_{p+n} + k \\ F_q + k & F_{q+m} + k & F_{q+n} + k \\ F_r + k & F_{r+m} + k & F_{r+n} + k \end{vmatrix}$$

for all integers p, q, r, m, and n.

(iii) We obtain from (3.1)

$$\begin{vmatrix} L_{n+3} & L_{n+2} & L_{n+1} & L_n \\ L_{n+2} & L_{n+3} & L_n & L_{n+1} \\ L_{n+1} & L_n & L_{n+3} & L_{n+2} \\ L_n & L_{n+1} & L_{n+2} & L_{n+3} \end{vmatrix}$$

$$= (L_{2n+4} + 3L_{2n+5})(L_{2n-2} + 3L_{2n-1})$$

$$= 25\, F_{2n+6} F_{2n}$$

$$= 25 \begin{vmatrix} F_{n+3} & F_{n+2} & F_{n+1} & F_n \\ F_{n+2} & F_{n+3} & F_n & F_{n+1} \\ F_{n+1} & F_n & F_{n+3} & F_{n+2} \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{vmatrix} .$$

(iv) We obtain from (4.1)

$$\begin{vmatrix} L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\ L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n+k} & L_{n+2k} & \cdots & L_n \end{vmatrix} = \frac{(L_n - L_{n+mk})^m - (-1)^{mk}(L_{nk} - L_{n+(m-1)k})^m}{1 - L_{mk} + (-1)^{mk}}$$

— a problem suggested by L. Carlitz [6].

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