RECREATIONAL MATHEMATICS
"DIFFERENCE SERIES" RESULTING FROM SIEVING PRIMES

JOSEPH S. MADACHY
4761 Bigger Road, Kettering, Ohio 45540

Sieving techniques are notoriously simple, and yet tedious, means of listing primes. By successively eliminating integers divisible by 2, 3, 5, 7, $11, \cdots, \sqrt{N}$, one has left all the primes less than N.

This paper will deal not with the list of integers which remain after each sieving procedure but with the differences between members of the remaining list of integers. The mathematical knowledge required to follow this material is strictly elementary (which may be one reason I continued working at it — if it had required more advanced mathematics, I might have drowned in a sea of mathematical notation).

If N consecutive integers are listed and all the even integers eliminated, a series of odd integers remains:

$$\text{(A)} \qquad 1,\ 3,\ 5,\ 7,\ 9,\ 11, \cdots .$$

The difference between each successive term in (A) is 2, and the number of integers in series (A) is $N/2$.

If now all integers in series (A) which are divisible by 3 are eliminated, the following series of integers remains:

$$\text{(B)} \qquad 1,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 25,\ 29,\ 31,\ 35,\ \cdots .$$

The number of integers in this series is $2N/6$ (fractions are deliberately not reduced since the numerators and denominators are important as we shall see). The differences between successive terms in series (B) is now

$$4,\ 2,\ 4,\ 2,\ 4,\ 2,\ 4,\ 2,\ 4,\ 2,\ 4, \cdots ,$$

which shows an obvious period of two terms, 4, 2. The sum of the members of this period is 6, or 2 x 3. Recall that series (B) was produced by eliminating integers divisible by 2 and 3.

Eliminating integers divisible by 5 from series (B) produces

(C) 1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, ⋯ .

The number of integers in this series is $8\underline{N}/30$, and the differences between successive terms in series (C) are

$$6,\ 4,\ 2,\ 4,\ 2,\ 4,\ 6,\ 2,\ 6,\ 4,\ 2,\ 4,\ 2,\ 4,\ 6,\ 2,\ 6,\cdots$$

which has an eight-term period

$$6,\ 4,\ 2,\ \textcircled{4},\ 2,\ 4,\ 6,\ 2\ .$$

(The circled integer will be explained later.) The sum of the members of this period is 30, or $2 \times 3 \times 5$.

Bear with me for one more round and note what happens when integers from series (C) which are divisible by 7 are eliminated:

(D)
1, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59,
61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 121,
127, 131, 137, 139, 143, 149, 151, 157, 163, 167, 169, 173, 179, 181,
187, 191, 193, 197, 199, 209, 211, 221, 223, 227, 229, 233, 239, 241,⋯

The number of integers in this series is $48\underline{N}/210$, and the difference series derived from series (D) has 48 terms in its period:

$$10,\ 2,\ 4,\ 2,\ 4,\ 6,\ 2,\ 6,\ 4,\ 2,\ 4,\ 6,\ 6,\ 2,\ 6,\ 4,$$
$$2,\ 6,\ 4,\ 6,\ 8,\ 4,\ 2,\ \textcircled{4},\ 2,\ 4,\ 8,\ 6,\ 4,\ 6,\ 2,\ 4,$$
$$6,\ 2,\ 6,\ 6,\ 4,\ 2,\ 4,\ 6,\ 2,\ 6,\ 4,\ 2,\ 4,\ 2,\ 10,\ 2\ .$$

The sum of these 48 terms is 210, or $2 \times 3 \times 5 \times 7$.

The difference series developed above have a number of intriguing properties. Readers undoubtedly will find more than I.

The first non-primes to appear in series (A), (B), (C), and (D) are 9, 25, 49, and 121, respectively. These are 3^2. 5^2. 7^2, and 11^2. If successive prime divisors are $p_1, p_2, p_3, p_4, \cdots$ where $p_1 = 2$, then the first non-prime in these series is p^2_{n+1}. If integers divisible by 11 are eliminated from series (D), the first non-prime will be 13^2, or 169 ($p_5 = 11$, $p_6 = 13$).

The sum of the terms in a difference series is $p_1 p_2 p_3 p_4 \cdots p_n$. The next difference series, obtained from series (D) by eliminating integers divisible by 11, would have a sum of $2 \times 3 \times 5 \times 7 \times 11$, or 2310. The number of terms in the difference series can also be calculated. Note that the number of integers remaining in the four series (A), (B), (C), and (D) are determined as follows:

(A): $$\frac{N}{2}$$

(B): $$\frac{N}{2} - \left(\frac{N}{2}\right)\left(\frac{1}{3}\right) = \frac{2N}{6}$$

(C): $$\frac{2N}{6} - \left(\frac{2N}{6}\right)\left(\frac{1}{5}\right) = \frac{8N}{30}$$

(D): $$\frac{8N}{30} - \left(\frac{8N}{30}\right)\left(\frac{1}{7}\right) = \frac{48N}{210} .$$

Going on to series (E) (left for readers with patience to develop), we have

$$\frac{48N}{210} - \left(\frac{48N}{210}\right)\left(\frac{1}{11}\right) = \frac{480N}{2310}$$

terms remaining. Generally, the number of integers remaining from the first <u>N</u> integers after elimination of integers divisible by successive $p_1, p_2, p_3, p_4, \cdots, p_n$, is:

$$\frac{(p_2 - 1)(p_3 - 1)(p_4 - 1)\cdots(p_n - 1)N}{p_1\, p_2\, p_3\, p_4 \cdots p_n} .$$

The number of terms in the resulting difference series is

$$(p_2 - 1)(p_3 - 1)(p_4 - 1)\cdots(p_n - 1) .$$

Each difference series is derived from the first $(p_1 p_2 p_3 p_4 \cdots p_n) + 1$ integers in the original set of N consecutive integers. The last term in the period of any difference series is 2. The first term in the period of any difference series is $p_{n+1} + 1$.

If d is the number of terms in a difference series $(d > 2)$ then the $d/2^{th}$ term is of special interest. These are the circled terms in the difference series shown perviously. If we ignore the final 2, I conjecture that the difference series are symmetrical about the $d/2^{th}$ term. I also think that this $d/2^{th}$ term will always be 4.

Since we can calculate the number of terms in a difference series, the point of symmetry, and the last term, we can write a complete period of any difference series by sieving only half as many integers. However, as we successively eliminate integers, the work will still be rather prohibitive. Going to $p_6 = 13$, the resulting difference series has 5760 terms, at $p_7 = 17$ there will be 92,160 terms. Perhaps other relationships within or between difference series will be found by readers — and reduce the labor involved.

The sum of the terms in one period of a difference series seems to be directly related to the differences between primes which are members of arithmetical progressions of primes. Karst* lists all arithmetical progressions of primes with 12 to 16 terms, and the difference between primes in a progression is often the sum of a difference series.

Difference series have been used to quite a limited extent in computer searches for primes. Except for 2, all even numbers are eliminated by simply adding 2 to the previous odd number. This automatically saves the computer half the work of searching for primes. Only one out of every two, or fifty percent, of the integers from 1 to N need be examined for primality.

If the second difference series above (4,2) is used, only 33% of the integers from 1 to N need be examined. The third difference series (6, 4, 2, 4, 2, 4, 6, 2) would reduce this to about 27%; the next difference series (with 48 terms) would reduce this to about 23%; and the next two difference series (with 480 and 5760 terms, respectively) would reduce this to about 21% and 19%, respectively. Not much gain in search efficiency is achieved after the third difference series is exploited. Further study of these difference series is required.

*Edgar Karst, "12 to 16 Primes in Arithmetical Progression," Journal of Recreational Mathematics, Vol. 2, No. 4 (October 1969), p. .

* * * * *