ASSOCIATED ADDITIVE DECIMAL DIGITAL BRACELETS*

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A bracelet is one period of a simply periodic series considered as a closed sequence with terms equally spaced around a circle. Distances between terms may be measured in degrees or in spaces. A bracelet may be regenerated by starting at any arbitrary point to apply the generating law. A bracelet may be cut at any arbitrary point for straight line representation without loss of any properties.

A digital bracelet may be constructed by starting with a pair of digits, affixing the units' digit of their sum, again affixing the units' digit of the sum of the last two digits, and continuing the process. (Some bracelets generated from a sequence of four digits have been discussed previously [1], [2]). This is equivalent to using the recurrence formula $u_{n+2} = u_n + u_{n+1}$ and, in the decimal system, reducing each sum modulo 10. When all operations of addition and multiplication are reduced modulo 10, the computations are in a modular arithmetic dealing with individual digits. In it 1, 7, 9, 3 and 2, 6, 8, 4 and their reverses are cyclic geometric progressions (G. P.) which themselves are bracelets.

The 60-digit [3] Fibonacci bracelet (F), so called since it is one period of the units' digits of the Fibonacci series, is generated by $u_1 = 0$, $u_2 = 1$. Five associated bracelets [4] result from other generating pairs, as follows:

A(1): 0 0 ' 0 0	B(3): 0 5 5 ' 0 5	C(4): 2 6 8 4 ' 2 6

D(12): 1 3 4	E(20): 0 2 2 4 6
7 1 8	0 6 6 2 8
9 7 6	0 8 8 6 4
3 9 2 ' 1 3	0 4 4 8 2 ' 0 2

F(60):	0 1 1 2 3	5 8 3 1 4	5 9 4 3 7
	0 7 7 4 1	5 6 1 7 8	5 3 8 1 9
	0 9 9 8 7	5 2 7 9 6	5 1 6 7 3
	0 3 3 6 9	5 4 9 3 2	5 7 2 9 1 ' 0 1

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The number in parentheses following the identifying letter is the number of ordered digit pairs in the bracelet. Except in A, this is the same as the number of digits in the bracelet — the length of the bracelet. Together, the six associated bracelets contain all 10^2 ordered digit pairs.

PROPERTIES OF ASSOCIATED BRACELETS

1. The number of digits in each bracelet is a factor of 60. This is in accord with the observation by Wall [5] that the length of any digital period resulting from $u_0 \neq 0$, $u_1 \neq 1$ divides the length of the Fibonacci digital period.

2. The sum of diametrically opposite digits is zero. Hence, the sum of the digits at the vertices of any inscribed polygon with an even number of sides is zero. The sum of the digits in every bracelet is zero.

3. The digits of C form a cyclic G. P. As the bracelets are arranged, the same G. P. appears in four of the digit columns of E and in the sums of its four rows. The reverse G. P. appears in 1 column of D, 4 columns of F, the sums of the four rows of D, the sums of the four rows of F, the sums of the rows of the last pentad column of F, and when the rows of F are broken up into triads in the sums of the rows of four of the five triad columns.

The cyclic G. P., 1, 7, 9, 3, appears in 2 columns of D, 8 columns of F, and the sums of the rows of the first two pentad columns of F.

The trivial G. P. of 0's which matches A, appears in the remaining digit column of E, and in the sums of the rows of the second column of triads in F. The remaining digit columns leading the pentad columns in F are in trivial G. P.'s of 0's, 5's, and 5's, which match B horizontally.

4. Bracelets D and F may be written in the terms:

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D: 1 3 4 7
   1 8 9 7
   6 3 9 2 ' 1 3
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F: 0 1 1 2 3   5 8 3 1 4   5 9 4 3 7   0 7 7 4 1
   5 6 1 7 8   5 3 8 1 9   0 9 9 8 7   5 2 7 9 6
   5 1 6 7 3   0 3 3 6 9   5 4 9 3 2   5 7 2 9 1 ' 0 1
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The sums of successive rows of D and of F are 5, 5, 0, a cyclic permutation (c.p.) of 8. The digit columns of D and F and the sums of the rows of the pentad columns in F are c.p.'s of the sets obtained by separately adding the digits of C to the digits of B. The digits in each of these sets are congruent modulo 5.

5. Bracelets E and F may be written in the forms:

E:	0 2 2 4	F: 0 1 1 2	3 5 8 3	1 4 5 9
	6 0 6 6	4 3 7 0	7 7 4 1	5 6 1 7
	2 8 0 8	8 5 3 8	1 9 0 9	9 8 7 5
	8 6 4 0	2 7 9 6	5 1 6 7	3 0 3 3
	4 4 8 2 ' 0 2	6 9 5 4	9 3 2 5	7 2 9 1 ' 0 1

Each row of E sums to 8 and each row of F sums to 2. Each quartet in the first column of F sums to 4, the other quartets sum to 9.

Each digit column consists of distinct even digits or of distinct odd digits. It follows that every even digit occurs with the same frequency in E. Also, in F the frequency of every even digit is the same, and the frequency of every odd digit is the same (and twice that of the even digits).

In E the digits in each column are in arithmetic progression with the successive common differences being 6, 8, 4, 2, a c.p. of C. In F the digits in each column are in arithmetic progression with the successive common differences being 4, 2, 6, 8, 4, 2, 6, 8, 4, 2, 6, 8, again involving a c.p. of C.

ASSOCIATION BY ADDITION

Each of the bracelets A, B, C, D, E, F is produced by the same recurrence formula. Consequently, proceeding clockwise, if the digits of the two bracelets are consecutively matched and added, the resulting sequence must follow the same formative law. But the six bracelets exhaust the 100-pair field, so the sequence created by the addition must duplicate all or part of one of the six bracelets.

The digits of each bracelet, as tabulated, are numbered consecutively from the left. In the additions a <u>selected</u> bracelet will be considered to be <u>operated on</u> by itself or by a shorter bracelet repeated. If the length of the selected bracelet is not a multiple of the length of the operator bracelet, then each is repeated to a total length equal to the l. c. m. of the two lengths.

The digits of the operator bracelet are successively matched with the initial digit of the selected bracelet in a series of additions. A particular addition is identified by the sequence number of the matching digit (m. d.) of the operator. The addition may produce a clockwise rotation (r.) of the selected bracelet or a change to another bracelet (b. c.).

Thus, when the third digit of D is matched with the initial digit of another D, the addition produces a b. c. into a series of 4 B's.

1 3 4 7	1 8 9 7	6 3 9 2	1 3 4 7	1 8 9 7	6 3 9 2
4 7 1 8	9 7 6 3	9 2 1 3	7 8 4 2	6 8 4 2	6 8 4 2
5'0 5 5 '	0 5 5'0	5 5 ' 0 0	7 1 8 9	7 6 3 9	2'1 3 4

and when C operates on D with the second digit of C matching the initial digit of D, the result of the addition is an r. of D through nine spaces or 270° from the position of the selected D.

Any bracelet operated on by A is not changed, which is equivalent to a rotation through 360°.

<u>B operated on</u> by B produces two rotations of B and one conversion to A's. Thus

B m. d.	r. or b. c.	B r.	Operator
1	A's	120°	B
2	120°	240°	B
3	240°	360°	A

<u>C operated on</u> by A, B, and C gives the following results:

C m. d.	r. or b. c.	C r.	Operator
1	90°	90°	C
2	180°	180°	C
3	A's	270°	C
4	270°	360°	A

When the 3-digit B operates on the 4-digit C, four B's and 3 C's are involved. The additions produce three D bracelets, with their initial digits 120° apart.

D operated on by A, B, C, and D produces:

B m.d.	r. or b.c.	D m.d.	r. or b.c.	D r.	Operator
1	240°	1	3 C's	30°	D
2	3 C's	2	300°	60°	D
3	120°	3	4 B's	90°	C
		4	3 C's	120°	B
		5	30°	150°	D
		6	60°	180°	C
		7	A's	210°	D
C m.d.	**r. or b.c.**	8	210°	240°	B
1	90°	9	150°	270°	C
2	270°	10	3 C's	300°	D
3	180°	11	4 B's	330°	D
4	4 B's	12	330°	360°	A

With reference to a fixed D, the initial digits of the C's generated by the D and B operators occur at 30° intervals, as do the initial digits of the B's produced by the D and C operators.

E operated on by A, B, C, D, and E gives:

C m.d.	r. or b.c.	E m.d.	r. or b.c.	E r.	Operator
1	216°	1	90°	18°	E
2	288°	2	324°	36°	E
3	144°	3	5 C's	54°	E
4	72°	4	54°	72°	C
		5	234°	90°	E
		6	180°	108°	E
		7	5 C's	126°	E
		8	126°	144°	C
		9	18°	162°	E

(continued on next p.)

E m.d.	r. or b.c.	E r.	Operator
10	36°	180°	E
11	A's	198°	E
12	198°	216°	C
13	162°	234°	E
14	252°	252°	E
15	5 C's	270°	E
16	270°	288°	C
17	306°	306°	E
18	108°	324°	E
19	5 C's	342°	E
20	342°	360°	A

With reference to a fixed E, the initial digits of the C's produced by the E operator are 18° apart. When D repeated five times operates on E repeated three times, F's in twelve different positions are produced. When B repeated twenty times operates on E repeated three times, F's in three different positions are produced. With reference to a fixed E, the F's produced by operator B are 120° apart, and those produced by operators B and D considered together are 24° apart.

F operated on by A, B, C, D, E, and F produces:

B m.d.	r. or b.c.
1	3 E's
2	240°
3	120°

C m.d.	r. or b.c.
1	144°
2	72°
3	216°
4	288°

(continued on next p.)

D m.d.	r. or b.c.
1	312°
2	336°
3	3 E's
4	24°
5	192°
6	3 E's
7	168°
8	264°
9	3 E's
10	96°
11	48°
12	3 E's

E m.d.	r. or b.c.
1	90°
2	342°
3	36°
4	5 D's
5	234°
6	270°
7	108°
8	5 D's
9	18°
10	198°
11	180°
12	5 D's
13	162°

E m. d.	r. or b. c.
14	126°
15	252°
16	20 B's
17	306°
18	54°
19	324°
20	5 D's

F operating on F produces thirty rotations of F, one A's, two 20 B's, four 15 C's, eight 5 D's, and three 15 E's. The rotations of F produced by all the operators neatly drop in at 6° intervals.

Beginning at 6°, the successive operators were:

F F E D F E F D E F F C F F E D F E F B E F F C F F E D F E
F D E F F C F F E B F E F D E F F C F F E D F E F D E F F A .

With reference to a fixed F, the initial digits of the B's produced by the various operators are 6° apart, as are the initial digits of the C's, the D's, and the E's produced by the various operators.

In general any one of the six associated bracelets with length p may be rotated through any desired multiple of 360°/p by operating on it with the proper associated bracelet of the same or shorter length and using the appropriate m. d.

Each of the operators produces rotations symmetrically distributed about 180°, so that when all the operator letters of the successive rotations of a selected bracelet are listed, a palindromic sequence is formed.

When the various operators produce bracelets other than the one operated on, the initial digits of the derived bracelets of the same type are equally spaced, when referred to the initial digit of the selected bracelet.

When the length of a shorter bracelet does not divide that of a longer one (as in the cases of B and C, B and E, D and E), then the operation of the shorter bracelet on the longer one always produces a bracelet of length equal to the l. c. m. of the lengths of the two bracelets involved. With reference to a fixed position of the longer bracelet, initial digits of the bracelets produced by the addition are equally distributed around a circle.

ASSOCIATION BY MULTIPLICATION

Since multiplication is repeated addition, it follows that multiplication of any one of these digital bracelets by a positive digit will either rotate the bracelet or convert it into another.bracelet. The rotations and conversions of the bracelets in the first column upon multiplication by the digits at the heads of the columns are shown in the body of the following table.

	8	4	2	6	3	9	7	1	5
B	A	A	A	A	B	B	B	B	B
C	90°	180°	270°	360°	90°	180°	270°	360°	A
D	C	C	C	C	270°	180°	90°	0°	B
E	90°	180°	270°	360°	90°	180°	270°	360°	A
F	E	E	E	E	270°	180°	90°	0°	B

The C's produced by multiplying D by 8, 4, 2, 6 in order go into each other by counterclockwise 90° rotations. The E's produced from F behave similarly.

To indicate that multiplication by 9 rotates a bracelet through 180° is equivalent to saying that the diametrically opposite digits sum to zero.

REFERENCES

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