ADVANCED PROBLEMS AND SOLUTIONS
Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

H-162 Proposed by David A. Klarner, University of Alberta, Edmonton Alberta, Canada.

Suppose \( a_{ij} \geq 1 \) for \( i, j = 1,2,\cdots \), show there exists an \( x \geq 1 \) such that

\[
(-1)^n \begin{vmatrix}
 a_{11} - x & a_{12} & \cdots & a_{1n} \\
 a_{21} & a_{22} - x^2 & \cdots & a_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 a_{n1} & a_{n2} & \cdots & a_{nn} - x^n \\
\end{vmatrix} \leq 0
\]

for all \( n \).

H-163 Proposed by H. H. Ferns, Victoria, B. C., Canada.

Prove the following identities:

1. \[
\sum_{k=1}^{n} 2^{2k-2} L_k F_{k+3} = 2^{2n} F_{n+2}^2 - 1
\]

2. \[
5 \sum_{k=1}^{n} 2^{2k-2} F_k L_{k+3} = 2^{2n} L_{n+1}^2 - 1
\]
where \( F_n \) and \( L_n \) are the \( n \)th Fibonacci and \( n \)th Lucas numbers, respectively.

**H-164 Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.**

Generalize H-127 and find a recurrence relation for the product \( C_n = A_n(x)B_n(y) \), where \( A_n \) and \( B_n \) satisfy the general second-order recurrence equations:

\[
\begin{align*}
A_{n+1}(x) &= R(x)A_n(x) + S(x)A_{n-1}(x) \\
B_{n+1}(y) &= P(y)B_n(y) + Q(y)B_{n-1}(y),
\end{align*}
\]

\( n \geq 1 \) and \( A_0, A_1, B_0, B_1 \) arbitrary.

**H-165 Proposed by H. H. Ferns, Victoria, B. C., Canada.**

Prove the identity

\[
\sum_{i=1}^{n} \binom{n}{i} \frac{F_{ki}}{F_{k-2}^i} = \left( \frac{F_k}{F_{k-2}} \right)^n F_{2n} \quad (k \neq 2),
\]

where \( F_i \) denotes the \( i \)th Fibonacci number.

**SOLUTIONS**

**A BASIS OF FACT?**

**H-132 Proposed by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania.**

Let

\[
F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad \text{for } n > 0
\]

define the Fibonacci sequence. Show that the Fibonacci sequence is **not** a basis of order \( k \) for any positive integer \( k \); that is, show that **not** every positive integer can be represented as a sum of \( k \) Fibonacci numbers, where repetitions are allowed and \( k \) is a fixed positive integer.
Solution by the Proposer.

Assume \( \{ F_n \}_{i=1}^{\infty} \) is a basis of order \( k \), where \( k \) is some fixed positive integer. Then, in particular, for given \( n > 0 \), any positive integer \( r \leq F_n \) would have a representation in the form

\[
r = \sum_{i=1}^{k} F_{n_i} ,
\]

where \( n_1 \leq n_2 \leq \cdots \leq n_k \) and \( n_k \leq n \). But the maximum number of distinct integers which could be formed by the right-hand side of (1) is clearly \( \leq n^k \).

Thus each of the \( F_n \) integers \( 1, 2, 3, \cdots, F_n \) would have to be expressed in a form capable of representing at most \( n^k \) distinct integers. Since, by choosing \( n \) large enough, we can make \( F_n > n^k \), a contradiction is obtained for the value of \( k \) under consideration. [The inequality \( F_n > n^k \) follows from the fact that \( F_n \) is approximately \( \frac{a^n}{\sqrt{5}} \) for large \( n \), where \( a = (1 + \sqrt{5})/2 \).]

SUM SHINE

H-133 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Characterize the sequences

i. \[ F_n = u_n + \sum_{j=1}^{n-2} u_j \]

ii. \[ F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j \]

iii. \[ F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-6} \sum_{j=1}^{m} \sum_{j=1}^{i} u_j \]

by finding starting values and recurrence relations. Generalize.
We shall first prove the iii part.

\[ F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-6} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j \]

\[ F_{n-1} = u_{n-1} + \sum_{j=1}^{n-3} u_j + \sum_{i=1}^{n-5} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-7} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j \]

\[ F_{n-2} = u_{n-2} + \sum_{j=1}^{n-4} u_j + \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j + \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j . \]

Since \( F_n - F_{n-1} - F_{n-2} = 0 \), we have

\[ 0 = (u_n - u_{n-1} - u_{n-2}) + \left( u_{n-2} - \sum_{j=1}^{n-4} u_j \right) \]

\[ + \left( \sum_{j=1}^{n-4} u_j - \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j \right) + \left( \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j - \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j \right) . \]

Cancelling out the terms, we get

\[ u_n = u_{n-1} + \sum_{m=1}^{n-8} \sum_{i=1}^{m} \sum_{j=1}^{i} u_j \].

(ii) Proceeding as above, we shall get

\[ u_n = u_{n-1} + \sum_{i=1}^{n-6} \sum_{j=1}^{i} u_j . \]

(i) Proceeding as above we shall get

\[ u_n = u_{n-1} + \sum_{j=1}^{n-4} u_j . \]
Generalization. If

\[ F_n = u_n + \sum_{j=1}^{n-2} u_j + \sum_{i=1}^{n-4} \sum_{j=1}^{i} u_j + \cdots + \]

\[ + \sum_{s=1}^{n-2r} \sum_{q=1}^{s} \sum_{p=1}^{q} \cdots \sum_{i=1}^{r} u_j \]

(r summations)

then proceeding as above, we shall get

\[ u_n = u_{n-1} + \sum_{s=1}^{n-2r-2} \sum_{q=1}^{s} \sum_{p=1}^{q} \cdots \sum_{i=1}^{r} u_j \]

Editorial Note: Professor Hoggatt obtained the solutions:

i) \( u_n = u(n; 2, 2) \)

ii) \( u_n = u(n; 3, 3) \)

iii) \( u_n = u(n; 4, 4) \) where \( u(n; p, q) \) represents the generalized Fibonacci number.

See V. C. Harris and C. C. Styles, "A Generalization of Fibonacci Numbers,"

CIRCLE TO THE RIGHT

\[ H-134 \text{ Proposed by L. Carlitz, Duke University, Durham, North Carolina.} \]

Evaluate the circulants

\[
\begin{bmatrix}
F_n & F_{n+k} & \cdots & F_{n+(m-1)k} \\
F_{n+(m-1)k} & F_n & \cdots & F_{n+(m-2)k} \\
\vdots & \vdots & \ddots & \vdots \\
F_{n+k} & F_{n+2k} & \cdots & F_n
\end{bmatrix}
\begin{bmatrix}
L_n & L_{n+k} & \cdots & L_{n+(m-1)k} \\
L_{n+(m-1)k} & L_n & \cdots & L_{n+(m-2)k} \\
\vdots & \vdots & \ddots & \vdots \\
L_{n+1} & L_{n+2k} & \cdots & L_n
\end{bmatrix}
\]

Solution by the Proposer.

We recall that
Hence if we put

\[ \Delta_m(F) = \begin{bmatrix} F_n & F_{n+k} & \cdots & F_{n+m-1k} \\ F_{n+m-1k} & F_n & \cdots & F_{n+(m-2)k} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+2k} & F_{n+2k} & \cdots & F_n \end{bmatrix}, \]

and

\[ \Delta_m(L) = \begin{bmatrix} L_n & L_{n+k} & \cdots & L_{n+m-1k} \\ L_{n+m-1k} & L_n & \cdots & L_{n+(m-2)k} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+2k} & L_{n+2k} & \cdots & L_n \end{bmatrix}, \]

we have

\[ \Delta_m(F) = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} F_{n+sk} \omega^{rs}, \quad \Delta_m(L) = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} L_{n+sk} \omega^{rs}. \]

Put

\[ \alpha = \frac{1 + \sqrt{5}}{2}, \quad \beta = \frac{1 - \sqrt{5}}{2}, \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n. \]

Then

\[
\sum_{s=0}^{m-1} F_{n+sk} \omega^{rs} = \frac{1}{\alpha - \beta} \sum_{s=0}^{m-1} (\alpha^{n+sk} - \beta^{n+sk}) \omega^{rs}
\]

\[
= \frac{1}{\alpha - \beta} \left\{ \frac{\alpha^n - \beta^n}{1 - \omega^r \alpha^k} - \frac{\beta^n - \alpha^n}{1 - \omega^r \beta^k} \right\}
\]

\[
= \alpha^n (1 - \alpha^{mk})(1 - \omega^r \beta^k) - \beta^n (1 - \beta^{mk})(1 - \omega^r \alpha^k)
\]

\[
\frac{1}{(\alpha - \beta)(1 - \omega^r \alpha)(1 - \omega^r \beta)}
\]

\[
= \frac{\alpha - \beta - (\alpha^{n+mk} - \beta^{n+mk}) - \omega^r (\alpha^{n+mk} - \beta^{n+mk})}{(\alpha - \beta)(1 - \omega^r \alpha)(1 - \omega^r \beta)}
\]

\[
= \frac{F_n - F_{n+mk} - (-1)^k \omega^r (F_{n-k} - F_{n+(m-1)k})}{(1 - \omega^r \alpha^k)(1 - \omega^r \beta^k)}
\]
so that

\[
\prod_{r=0}^{m-1} \sum_{s=0}^{m-1} F_{n+sk} \omega^{rs} = \prod_{r=0}^{m-1} \sum_{s=0}^{m-1} \frac{F_n - F_{n+mk} - (-1)^k \omega^r (F_{n-k} - F_{n+(m-1)k})}{(1 - \omega^r \alpha^k)(1 - \omega^r \beta^k)}
\]

\[
= \frac{(F_n - F_{n+mk})^m - (-1)^mk(F_{n-k} - F_{n+(m-1)k})^m}{(1 - \alpha^{mk})(1 - \beta^{mk})}
\]

Therefore

\[
(*) \quad \Delta_m(F) = \frac{(F_n - F_{n+mk})^m - (-1)^mk(F_{n-k} - F_{n+(m-1)k})^m}{1 + (-1)^mk - L_{mk}}
\]

Similarly,

\[
\sum_{s=0}^{m-1} L_{n+sk} \omega^{rs} = \sum_{s=0}^{m-1} (\alpha^{n+sk} + \beta^{n+sk}) \omega^{rs}
\]

\[
= \alpha^n \frac{1 - \alpha^{mk}}{1 - \omega^r \alpha^k} + \beta^n \frac{1 - \beta^{mk}}{1 - \omega^r \beta^k}
\]

\[
= \frac{\alpha^n + \beta^n - \alpha^{n+mk} - \beta^{n+mk} - \omega^r (\alpha^{n+mk} + \beta^{n+mk})}{(1 - \omega^r \alpha^k)(1 - \omega^r \beta^k)}
\]

\[
= \frac{L_n - L_{n+mk} - (-1)^k \omega^r (L_{n-k} - L_{n+(m-1)k})}{(1 - \omega^r \alpha^k)(1 - \omega^r \beta^k)}
\]

It follows that

\[
(**) \quad \Delta_m(L) = \frac{(L_n - L_{n+mk})^m - (-1)^mk(L_{n-k} - L_{n+(m-1)k})^m}{1 + (-1)^mk - L_{mk}}
\]

Also solved by D. Jaiswal (India).
THE GREATEST INTEGER!

H-135 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida.

PART I

Show that

\[ j + 1 = \sum_{d=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \binom{j - d}{d} 2^{j-2d} (-1)^d \]

where \( j \geq 0 \) and \( \left\lfloor \frac{j}{2} \right\rfloor \) is the greatest integer not exceeding \( j/2 \).

PART II

Show that

\[ F_{(j+1)n} = \sum_{d=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \binom{j - d}{d} \binom{n}{d} 2^{j-2d} (-1)^{n+1} \]

where \( j \geq 0 \) and \( \left\lfloor \frac{j}{2} \right\rfloor \) is the greatest integer not exceeding \( j/2 \).

Solution by the Proposer.

PART I

We have (see "A Generalization of the Connection between the Fibonacci Sequence and Pascal's Triangle," by Joseph A. Raab, this quarterly, Vol. 1, No. 3, October 1963, pp. 25-26) that

\[ \sum_{d=0}^{\left\lfloor \frac{j}{2} \right\rfloor} \binom{j - d}{d} 2^{j-2d} (-1)^d = x_j \]

and \( x_{j+2} = 2x_{j+1} - x_j \) for all \( j \geq 0 \). Let \( S \) be the set of all integers \( (j + 1) \geq 0 \) for which the theorem is true, \( 1 = x_q \) and \( 2 = x_1 \), so 1 and 2 are in \( S \). Suppose \( q \) and \( q + 1 \) are in \( S \), so that \( q = x_{q-1} \) and \( q + 1 = x_q \). Then
\[ x_{q+1} = 2x_q - x_{q-1} = 2(q + 1) - q = q + 2. \]

Thus \( q + 2 \) is in the set \( S \) and the proof is complete by mathematical induction.

**PART II**

The same reference as given in Part I yields the result that

\[
\left\lfloor \frac{j}{3} \right\rfloor \sum_{d=0}^{\left\lfloor \frac{j}{d} \right\rfloor} \binom{j}{d} (-1)^{d+1} j^{(n+1)d} = x_j
\]

and

\[ x_{j+2} = L_n x_{j+1} + (-1)^{n+1} x_j \]

for all \( j \geq 0 \). Let \( S \) be the set of all integers \( (j + 1) > 0 \) for which the theorem is true. \( F_n = F_n x_q \) and \( F_{2n} = F_n L_n = F_n x_1 \), so \( 1 \) and \( 2 \) are in \( S \). Suppose \( q \) and \( q + 1 \) are in \( S \), so that \( F_q = F_n x_q \) and \( F_{(q+1)n} = F_n x_q \). Then

\[
F_n x_{q+1} = F_n L_n x_q + F_n (-1)^{q+1} x_{q-1} = L_n F_{(q+1)n} + (-1)^{n+1} F_q = F_{(q+2)n}
\]

by a known identity (see "Some Fibonacci Results Using Fibonacci-Type Sequences," by I. Dale Ruggles, this quarterly, Vol. 1, No. 2, April, 1963, p. 77). Thus \( q + 2 \) is in the set \( S \) and the proof is complete by mathematical induction.

Also solved by B. King, L. Carlitz, D. Jaiswal (India), and D. Zeitlin.

**SQUEEZE PLAY**

H-136 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California, and D. A. Lind, University of Virginia, Charlottesville, Va.

Let \( \{H_n\} \) be defined by \( H_1 = p, H_2 = q, H_{n+2} = H_{n+1} + H_n \) \((n \geq 1)\)
where \( p \) and \( q \) are non-negative integers. Sh there are integers \( N \) and
k such that $F_{n+k} < H_n \leq F_{n+k+1}$ for all $n > N$. Does the conclusion hold if $p$ and $q$ are allowed to be non-negative reals instead of integers?

*Solution by Gerald A. Edgar, Student, University of California, Santa Barbara, California.*

In order for the result to be true, we must have $p > 0$ or $q > 0$. Let

$$a = \left(1 + \sqrt{5}\right)/2, \quad b = \left(1 - \sqrt{5}\right)/2.$$  

Define $f(n) = \lfloor an + 1/2 \rfloor$, for $n$ a positive integer, where $\lfloor x \rfloor$ is the greatest integer in $x$ (thus $f(n)$ is the nearest integer to $an$). We now prove that $f(f(n)) = f(n) + n$. The definition of $f$ gives

1. $an + \frac{1}{2} \geq f(n) > an - \frac{1}{2}$

2. $af(n) + \frac{1}{2} > f(f(n)) > af(n) - \frac{1}{2}$ .

But (1) is the same as

$$\frac{f(n)}{a} + \frac{1}{2a} > n \geq \frac{f(n)}{a} - \frac{1}{2a},$$

or, since $(1/a) = a - 1$,

$$(a - 1)f(n) + (a - 1)/2 > n \geq (a - 1)f(n) - (a - 1)/2$$

or

3. $af(n) + \frac{a}{2} - \frac{1}{2} > n + f(n) \geq af(n) - \frac{a}{2} + \frac{1}{2}.$

Equations (2) and (3) give

$$\frac{a}{2} > f(n) + n - f(f(n)) \geq -\frac{a}{2}.$$  

But $a/2 < 1$, and $f(n) + n - f(f(n))$ is an integer, so it must be zero, and we have
Because of its recurrence, $H_n$ must have the form $H_n = cn^m + db^n$ for some constants $c$ and $d$. Now $|b| < 1$, so

$$
\lim_{n \to \infty} (aH_n - H_{n+1}) = \lim_{n \to \infty} (cn^{m+1} + db^{n+1} - cn^m - db^n) = \lim_{n \to \infty} \sqrt{5}dn = 0
$$

Thus there is an integer $N$ such that $|aH_n - H_{n+1}| < \frac{1}{2}$ for all $n \geq N$. In particular, $|aH_N - H_{N+1}| < \frac{1}{2}$, so, since $H_{N+1}$ is an integer,

$$
H_{N+1} = \lfloor aH_N + \frac{1}{2} \rfloor = f(H_N)
$$

It is now an easy induction to show that

(5) $H_{N+m} = f^m(H_N)$

for $m = 0, 1, 2, \cdots$, where $f^m$ is the $m$th iterate of $f$ defined by

$$
\begin{align*}
&f^0(x) = x \\
&f^{n+1}(x) = f(f^n(x))
\end{align*}
$$

(Note that in particular, $f(F_n) = F_{n+1}$ for $n = 2, 3, \cdots$ for the Fibonacci numbers.) Since $H_N$ is a positive integer, there is an integer $k$ such that

(6) $F_{N+k} < H_N < F_{N+k+1}$

We may then obtain by induction (using the fact that $f$ is strictly increasing on the positive integers)

$$
F_{n+k} < H_n < F_{n+k+1}
$$

for all $n \geq N$. 
The result does not hold for non-negative reals in general; take

\[ p = a/\sqrt{3}, \quad q = a^2/\sqrt{3}; \]

then \( H_n > F_n \) when \( n \) is even and \( H_n < F_n \) when \( n \) is odd.

* * * *

Also solved by J. Desmond, A. Shannon, and M. Yoder.