ORTHOGONAL EXPANSION DERIVED FROM THE EXTREME VALUE DISTRIBUTION

J. C. AHUJA
Portland State College, Portland, Oregon

1. INTRODUCTION

The cumulative distribution function, $F(x)$, of the extreme value distribution is given by

$$F(x) = e^{-e^{-x}}, \quad \text{for} \quad -\infty < x < \infty, \tag{1}$$

and the density function, $f(x) = F'(x)$, is obtained as

$$f(x) = e^{-(x+e^{-x})}, \quad \text{for} \quad -\infty < x < \infty. \tag{2}$$

The extreme value distribution has found a number of applications. Cramer [2] derives (2) as an asymptotic density of the first value from the top for certain transformed variates in a random sample of n observations drawn from Laplace's and normal distributions. The distribution function (1) was first used by Gompertz [3] in connection with actuarial life tables and later on has been used extensively in the study of growth.

The purpose of this paper is (i) to find an explicit expression for the moment generating function of the standardized extreme value distribution and (ii) to derive an orthogonal expansion (Type A series) from the extreme value density in a manner similar to the way in which Gram [4] and Charlier [1] derived an orthogonal expansion from the normal density by making use of the Hermite polynomials which are orthogonal with respect to the normal density. The orthogonal expansion requires the calculation of first eight standardized moments of (2) which in turn involve the evaluation of the Riemann zeta function. This difficulty is overcome by using the tabular values of the Riemann zeta function given by Steiljes [6].

2. MOMENT GENERATING FUNCTION

The moment generating function, $M_x(u)$, of the density function $f(x)$ is

$$M_x(u) = \int_{-\infty}^{\infty} e^{ux} e^{-(x+e^{-x})} dx \quad ,$$

which, on substituting $s = e^{-x}$, becomes

$$(3) \qquad M_x(u) = \int_0^{\infty} s^{-u} e^{-s} ds$$

$$= \Gamma(1 - u)$$

$$= \sum_{k=0}^{\infty} \Gamma^{(k)}(1)(-u)^k/k!$$

where $\Gamma^{(k)}(1)$ is the k^{th} derivative of the gamma function, $\Gamma(p)$, at $p = 1$. This proves the following.

Lemma 1. The moment generating function of the extreme value density $f(x)$ is given by (3).

According to Jordan [5], the n^{th} derivative of $\Gamma(p)$ at $p = 1$ is

$$(4) \qquad \Gamma^{(n)}(1) = (-1)^n \sum \frac{n!}{d_1! \, d_2! \cdots d_n!} C^{d_1} (S_2/2)^{d_2} \cdots (S_n/n)^{d_n} \quad ,$$

where the summation is over non-negative integers $d_1, d_2, \cdots, d_n$ such that $d_1 + 2d_2 + 3d_3 + \cdots + nd_n = n$; S_k is the Riemann zeta function defined by

$$S_k = \sum_{n=1}^{\infty} n^{-k} \quad ,$$

and C is Euler's constant which, correct to nine decimal places, is 0.577215665^{-}.

If μ_1' and μ_2 denote the mean and variance of $f(x)$, then (3) and (4) give us

$$\mu_1' = C \quad \text{and} \quad \mu_2 = S_2 .$$

Defining $z = (x - C)/\sqrt{S_2}$, we get the standardized extreme value density function

$$g(z) = \sqrt{S_2}\, e^{-[C + \sqrt{S_2} z + e^{-(C + \sqrt{S_2} z)}]}, \quad \text{for } -\infty < z < \infty \qquad (5)$$

The moment generating function, $M_z(u)$, of $g(z)$ is obtained as

$$\begin{aligned} M_z(u) &= E(e^{uz}) \\ &= e^{-Cu/\sqrt{S_2}} M_x(u/\sqrt{S_2}) , \end{aligned}$$

which, by Lemma 1, becomes

$$\begin{aligned} M_z(u) &= \left[\sum_{h=0}^{\infty} (C/\sqrt{S_2})^h (-u)^h/h!\right]\left[\sum_{k=0}^{\infty} \Gamma^{(k)}(1)(-u/\sqrt{S_2})^k/k!\right] \\ &= \sum_{r=0}^{\infty} \alpha_r u^r/r! \quad , \end{aligned} \qquad (6)$$

where α_r is the r^{th} standardized moment of $g(z)$ and

$$\alpha_r = \sum_{j=0}^{r} (-1)^r (1/S_2)^{r/2} \binom{r}{j} C^{r-j} \Gamma^{(j)}(1) . \qquad (7)$$

This completes the proof of the following:

Theorem 1. The moment generating function of the standardized extreme value distribution $g(z)$ is given by (6).

The first eight of the expressions in (7), using (4), are

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_2 &= 1 \\ \alpha_3 &= 2S_3/\sqrt{S_2^3} \end{aligned}$$

$$\alpha_4 = (3S_2^2 + 6S_4)/S_2^2$$

$$\alpha_5 = (20S_2S_3 + 24S_5)/\sqrt{S_2^5}$$

$$\alpha_6 = (15S_2^3 + 40S_3^2 + 90S_2S_4 + 120S_6)/S_2^3$$

$$\alpha_7 = (210S_2^2S_3 + 420S_3S_4 + 504S_2S_5 + 720S_7)/\sqrt{S_2^7}$$

$$\alpha_8 = (105S_2^4 + 1120S_2S_3^2 + 1260S_2^2S_4 + 1260S_4^2 + 2688S_3S_5 + 3360S_2S_6 + 5040S_8)/S_2^4 \; .$$

The values of S_k for $k = 2, 3, \cdots, 70$ have been computed by Stieltjes [6] up to 32 decimal places. Using his tabular values, we have

$S_2 = 1.644934067$	$S_6 = 1.017343062$
$S_3 = 1.202056903$	$S_7 = 1.008349277$
$S_4 = 1.082323234$	$S_8 = 1.004077356$
$S_5 = 1.036927755^+$	

The substitution of S's give the numerical values of α's as

$\alpha_1 = 0.000000000$	$\alpha_5 = 18.566615980$
$\alpha_2 = 1.000000000$	$\alpha_6 = 91.414247335^-$
$\alpha_3 = 1.139547099$	$\alpha_7 = 493.149891500$
$\alpha_4 = 5.400000000$	$\alpha_8 = 3091.022943246$

3. ORTHOGONAL POLYNOMIALS

If α_r denotes the r^{th} standardized moment of g(z), then, according to Szego [7], the orthogonal polynomials $q_n(z)$ associated with the density function g(z) are given by

$$q_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} 1 & 0 & 1 & \alpha_3 & \cdots & \alpha_n \\ 0 & 1 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n+1} \\ & & \vdots & & & \\ \alpha_{n-1} & \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{2n-1} \\ 1 & z & z^2 & z^3 & \cdots & z^n \end{vmatrix} \tag{8}$$

where the leading coefficient of $q_n(z)$ is one and

$$D_n = \begin{vmatrix} 1 & 0 & 1 & \alpha_3 & \cdots & \alpha_n \\ 0 & 1 & \alpha_3 & \alpha_4 & \cdots & \alpha_{n+1} \\ & & \vdots & & & \\ \alpha_{n-1} & \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{2n-1} \\ \alpha_n & \alpha_{n+1} & \alpha_{n+2} & \alpha_{n+3} & \cdots & \alpha_{2n} \end{vmatrix} \tag{9}$$

The polynomials $q_n(z)$ have the orthogonality property that

$$\int_{-\infty}^{\infty} q_m(z)q_n(z)g(z)dz = \begin{cases} D_n/D_{n-1} & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \tag{10}$$

Substituting for α's in (8), the polynomials $q_n(z)$, correct to six decimal places, for $n = 0, 1, 2, 3,$ and 4, are obtained as

$$\begin{aligned} q_0(z) &= 1 \\ q_1(z) &= z \\ q_2(z) &= z^2 - 1.139547z - 1 \\ q_3(z) &= z^3 - 3.634938z^2 - 1.257817z + 2.495391 \\ q_4(z) &= z^4 - 7.557958z^3 + 6.560849z^2 + 14.769958z - 3.348201 \ . \end{aligned}$$

4. DERIVATION OF ORTHOGONAL EXPANSION

Suppose that a density function, $h(z)$, can be represented formally by an infinite series of the form

$$h(z) = g(z) \sum_{n=0}^{\infty} a_n q_n(z) \ , \tag{11}$$

where the $q_n(z)$ are orthogonal polynomials associated with the density function $g(z)$.

Multiplying both sides of (11) by $q_n(z)$ and integrating from $-\infty$ to ∞, we have, in virtue of the orthogonality relationship (10),

$$a_n = \frac{D_{n-1}}{D_n} \int_{-\infty}^{\infty} h(z) q_n(z) dz \quad . \tag{12}$$

The reader familiar with harmonic analysis will recognize the resemblance between this procedure and the evaluation of constants in a Fourier series.

The first five values of a's, given by (12), are computed as

$$a_0 = 1$$

$$a_1 = 0$$

$$a_2 = 0$$

$$a_3 = 0.0500572(\beta_3 - 1.139547)$$

$$a_4 = 0.0045512(\beta_4 - 7.557958\beta_3 + 3.212648)$$

where β_r is the r^{th} standardized moment of $h(z)$.

Substituting for the a's in (11), we have

Theorem 2. The orthogonal expansion (Type A series) derived from the standardized extreme value density $g(z)$ is

$$h(z) = g(z)[1 + 0.0500572(\beta_3 - 1.139547)q_3(z) + 0.0045512(\beta_4 - 7.557958\beta_3 + 3.212648)q_4(z) + \cdots]$$

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