SUMS OF POWERS OF FIBONACCI AND LUCAS NUMBERS

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1. Hunter has stated as a problem in this Quarterly [2] the identity

(1) $$F_{n-1}^4 + F_n^4 + F_{n+1}^4 = 2[2F_n^2 + (-1)^n]^2 .$$

This can be proved rapidly in the following way. In the identity

(2) $$x^4 + y^4 + (x + y)^4 = 2(x^2 + xy + y^2)^2 ,$$

take $x = F_{n-1}$, $y = F_n$. Then

$$F_{n-1}^4 + F_n^4 + F_{n+1}^4 = 2(F_{n-1}^2 + F_{n-1}F_n + F_n^2)^2 .$$

Since

$$F_{n-1}^2 + F_{n-1}F_n + F_n^2 = F_{n-1}F_{n+1} + F_n^2 = 2F_n^2 + (-1)^n ,$$

we immediately get (1).

Similarly if we take $x = L_{n-1}$, $y = L_n$ in (2), then since

$$L_{n-1}^2 + L_{n-1}L_n + L_n^2 = L_{n-1}L_{n+1} + L_n^2 = 2L_n^2 - 5(-1)^n ,$$

we get the companion formula

(3) $$L_{n-1}^4 + L_n^4 + L_{n+1}^4 = 2[2L_n^2 - 5(-1)^n]^2 .$$

In the same way the identities

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$$(x + y)^5 - x^5 - y^5 = 5xy(x + y)(x^2 + xy + y^2),$$

$$(x + y)^7 - x^7 - y^7 = 7xy(x + y)(x^2 + xy + y^2)^2 ,$$

lead to the following:

$$F_{n+1}^5 - F_n^5 - F_{n-1}^5 = 5F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n) , \tag{4}$$

$$L_{n+1}^5 - L_n^5 - L_{n-1}^5 = 5L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n), \tag{5}$$

$$F_{n+1}^7 - F_n^7 - F_{n-1}^7 = 7F_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2 , \tag{6}$$

$$L_{n+1}^7 - L_n^7 - L_{n-1}^7 = 7L_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2 . \tag{7}$$

Cauchy has proved (see [1, p. 31]) that if p is a prime 3 then

$$(x + y)^p - x^p - y^p = pxy(x + y)(x^2 + xy + y^2)f_p(x,y) , \tag{8}$$

where $f_p(x,y)$ is a polynomial with integral coefficients. For $p \equiv 1 \pmod 6$ there is the stronger result:

$$(x + y)^p - x^p - y^p = pxy(x + y)(x^2 + xy + y^2)^2 g_p(x,y) , \tag{9}$$

where $g_p(x,y)$ is a polynomial with integral coefficients. Substituting $x = F_{n-1}$, $y = F_n$, we get

$$F_{n+1}^p - F_n^p - F_{n-1}^p = pF_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)F_{n,p} ,$$

$$L_{n+1}^p - L_n^p - L_{n-1}^p = pL_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)L_{n,p} ,$$

where $F_{n,p}$ and $L_{n,p}$ are integers. If $p \equiv 1 \pmod 6$ we get

$$F_{n+1}^p - F_n^p - F_{n-1}^p = pF_{n+1}F_nF_{n-1}(2F_n^2 + (-1)^n)^2 F'_{n,p} ,$$

$$L_{n+1}^p - L_n^p - L_{n-1}^p = pL_{n+1}L_nL_{n-1}(2L_n^2 - 5(-1)^n)^2 L'_{n,p} ,$$

where $F'_{n,p}$ and $L'_{n,p}$ are integers.

2. To get more explicit results, we proceed as follows. Consider the identity

$$(10)\qquad \frac{x}{1 - xw} + \frac{y}{1 - yw} + \frac{z}{1 - zw} = \frac{(x + y + z) - 2(xy + xa + yz)w + 3xyzw^2}{1 - (x + y + z)w + (xy + xz + yz)w^2 - xyzw^3} .$$

We take $z = -x - y$. Then (10) becomes

$$(11)\qquad -\frac{x}{1 - xw} - \frac{y}{1 - yw} + \frac{x + y}{1 + (x + y)w} = \frac{-2Uw + 3Vw^2}{1 - Uw^2 + Vw^3} ,$$

where

$$(12)\qquad U = x^2 + xy + y^2, \quad V = xy(x + y) .$$

We have

$$(1 - Uw^2 + Vw^3)^{-1} = \sum_{r=0}^{\infty} w^{2r}(U - Vw)^r$$

$$= \sum_{r=0}^{\infty} w^{2r} \sum_{s=0}^{r} (-1)^s \binom{r}{s} U^{r-s}V^s w^s$$

$$= \sum_{k=0}^{\infty} (-1)^k w^k \sum_{r} \binom{r}{k - 2r} U^{3r-k}V^{k-2r} .$$

Since the left member of (11) is equal to

$$\sum_{k=0}^{\infty} [(-1)^k (x + y)^{k+1} - x^{k+1} - y^{k+1}] w^k ,$$

it follows that

$$(-1)^k(x + y)^{k+1} - x^{k+1} - y^{k+1}$$

$$= (-1)^k\, 2 \sum_r \binom{r}{k - 2r - 1} U^{3r-k+2}\, V^{k-2r-1}$$

$$+ (-1)^k 3 \sum_r \binom{r}{k - 2r - 2} U^{3r-k+2} V^{k-2r-1} .$$

Since

$$2\binom{r}{k - 2r - 1} + 3\binom{r}{k - 2r - 2} = \frac{k + 1}{k - 2r - 1}\binom{r}{k - 2r - 2} ,$$

we have

$$(x + y)^{k+1} - (-1)^k(x^{k+1} + y^{k+1})$$

(13)

$$= \sum_r \frac{k + 1}{k - 2r - 1}\binom{r}{k - 2r - 2} U^{3r-k+2} V^{k-2r-1}.$$

When k is odd, it is to be understood that for $r = (k - 1)/2$, the coefficient on the right is 2.

Replacing k by $2k$ in (13), we get

$$(x + y)^{2k+1} - x^{2k+1} - y^{2k+1}$$

$$= \sum_r \frac{2k + 1}{2k - 2r - 1}\binom{r}{2k - 2r - 2} U^{3r-2k+2} V^{2k-2r-1};$$

the range of r is determined by

(15) $$r < k, \quad 2k - 2 \leq 3r .$$

In particular (14) implies

(16)
$$(x + y)^{6k+1} - x^{6k+1} - y^{6k+1} = \sum_{r=0}^{k-1} \frac{6k + 1}{2r + 1} \binom{3k - r - 1}{2r} U^{3k-3r-1} V^{2r+1} .$$

For example, we have

$$(x + y)^7 - x^7 - y^7 = 7 U^2V$$
$$(x + y)^{13} - x^{13} - y^{13} = 13 U^2V(U^3 + 2V^2)$$
$$(x + y)^{19} - x^{19} - y^{19} = 19U^2V(U^6 + 7U^3V^2 + 3V^4) .$$

We also have from (14)

(17)
$$(x + y)^{6k+5} - x^{6k+5} - y^{6k+5} = \sum_{r=0}^{k} \frac{6k + 5}{2r + 1} \binom{3k - r + 1}{2r} U^{3k-3r+1} V^{2r+1} .$$

For example,

$$(x + y)^5 - x^5 - y^5 = 5UV$$
$$(x + y)^{11} - x^{11} - y^{11} = 11 UV(U^3 + V^2)$$
$$(x + y)^{17} - x^{17} - y^{17} = 17(UV(U^6 + 5U^3V^2 + V^4).$$

When $6k + 1$ is prime, the coefficients on the right of (16) are divisible by $6k + 1$; moreover the right member has the polynomial factor U^2. When $6k + 5$ is prime, the coefficients on the right of (17) are divisible by $6k + 5$; moreover the right member has the polynomial factor U. Thus (16) and (17) furnish explicit formulas for the factors $f_p(x,y)$ and $g_p(x,y)$ occurring in (8) and (9).

In addition we have the identity

(18)
$$(x + y)^{6k+3} - x^{6k+3} - y^{6k+3} = \sum_{r=0}^{k} \frac{6k + 3}{2r + 1} \binom{3k - r}{2r} U^{3k-3r} V^{2r+1} .$$

For example

$$(x + y)^9 - x^9 - y^9 = 9U^3V + 3V^3$$

$$(x + y)^{15} - x^{15} - y^{15} = 15U^6V + 50U^3V^3 + 3V^5 .$$

For even exponents we get

$$(x + y)^{2k} + x^{2k} + y^{2k}$$

(19)
$$= 2U^k + \sum_{0<3r<k} \frac{k}{r}\binom{k - r - 1}{sr - 1} U^{k-3r}V^{2r} .$$

In particular, (19) yields

$$(x + y)^{6k} + x^{6k} + y^{6k}$$

(20)
$$= 2U^{3k} + \sum_{r=1}^{k} \frac{3k}{r}\binom{3k - r - 1}{2r - 1} U^{3k-3r}V^{2r} .$$

The first few coefficients in the right member of (19) are given by the following table.

k \ r	0	1	2	3
1	2			
2	2			
3	2	3		
4	2	8		
5	2	15		
6	2	24	3	
7	2	35	14	
8	2	48	40	
9	2	63	90	3
10	2	80	175	20

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$$\lim_{n\to\infty} \frac{1}{N} A(N,j,m) = \frac{1}{m} \qquad \text{for} \qquad j = 0, 1, \cdots, n-1.$$

(see [5]).

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