ON THE COMPLETENESS OF THE LUCAS SEQUENCE
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It is well known* that the Lucas sequence

\[ L_0, \ L_1, \ L_2, \ \cdots = 2, \ 1, \ 3, \ \cdots \]

is complete. It is easy to see that if \( 0 \leq m < n \), the integer \( L_{n+1} - 1 \) can't be represented as a sum of distinct \( L_i \) with \( i \neq m, n \). Thus \( \{ L_j \} \) is not complete after the removal of two arbitrary terms \( L_m, L_n \). We will also show that the sequence is complete after the removal of any one term \( L_n \) with \( n \geq 2 \).

Let \( N \) be a positive integer. It is well known that \( N \) is a (maximal) sum of \( L_i \)'s, that is,

\[ (1) \quad N = L_{i_1} + L_{i_2} + \cdots + L_{i_\beta} \quad \text{with} \quad \begin{cases} i_1 \geq 0 \quad \text{and} \\ i_{\nu + 1} - i_\nu \geq 2 \quad \text{for} \quad 1 \leq \nu < \beta. \end{cases} \]

We suppose \( L_n \) is one of the terms in the representation (1), for otherwise we have nothing to show, say \( n = i_\alpha \leq i_\beta \). Then

\[ (2) \quad M = L_{i_1} + L_{i_2} + \cdots + L_{i_\alpha} \leq L_n + L_{n-2} + \cdots + L_k + L_0 \]

\[ = \begin{cases} L_{n+1} - 1 \quad \text{if} \quad n \text{ is even,} \\ L_{n+1} + 1 \quad \text{and} \quad k = 2 \quad \text{if} \quad n \text{ is odd.} \end{cases} \]

If \( M = L_{n+1} + 1 \), we replace the sum (2) for \( M \) by \( L_1 + L_{n+1} \) in (1). If \( M = L_{n+1} \) we replace the sum (2) for \( M \) by \( L_{n+1} \) in (1). Observe that \( L_{n+1} \) does not appear in (1). If \( M \leq L_{n+1} - 1 \), we can re-represent it as a sum of distinct terms \( L_i \) with \( 0 \leq i \leq n - 1 \), and so we are through in this final case.


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