REPRESENTATION OF NATURAL NUMBERS AS SUMS OF GENERALIZED FIBONACCI NUMBERS – II

D. E. DAYKIN
University of Malaya, Kuala Lumpur, Malaysia

The well-known observation of Zeckendorf is that every positive integer N has a unique representation

$$N = u_{i_1} + u_{i_2} + \cdots + u_{i_d} ,$$

where

(1) $$i_1 \geqq 1 \text{ and } i_{\nu+1} - i_\nu \geqq 2 \text{ for } i \leqq \nu < d ,$$

and $\{u_n\}$ is the Fibonacci sequence

$$\cdots, 0, 0, 1, 2, 3, 5, 8, 13, \cdots$$

defined by

(2) $$\begin{cases} u_n = 0 & \text{for } n \leqq 0 , \\ u_1 = 1, \ u_2 = 2, & \text{and} \\ u_{n+1} = u_n + u_{n-1} & \text{for } n \geqq 2 . \end{cases}$$

Existence of such a representation follows from (2), and its uniqueness follows easily from the identity

(3) $$u_{n+1} = 1 + u_n + u_{n-2} + u_{n-4} + \cdots \quad \text{for } n \geqq 0 .$$

The object of this note is to discuss very general methods for uniquely representing integers, of which Zeckendorf's theorem is a special case. I feel that my results give a fairly complete description of the representations; they certainly extend the treatment of an earlier paper of the same name [5].

Here are some remarks on the notation which will be followed throughout this paper. We reserve the brackets $\{\cdots\}$, $(\cdots)$ and $[\cdots]$ for sequences, vectors and matrices, respectively. By V we denote the set of all vectors $(i_1, i_2, \cdots, i_d)$ of various dimensions $d \geq 1$, whose components i_ν are integers with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_d$. Often we will write I instead of $(i_1, i_2, \cdots, i_d)$ and M instead of $[m_{\mu,\nu}]$. Also $\{a_n\}$, $n = 1, 2, 3, \cdots$ will denote any sequence of integers satisfying axiom 1.

Axiom 1. The sequence is strictly increasing and its first term is 1.

For convenience, we write $a(I)$ or $a(i_1, i_2, \cdots, i_d)$ for the number

$$a(I) = a(i_1, i_2, \cdots, i_d) = a_{i_1} + a_{i_2} + \cdots + a_{i_d} .$$

It will be noted that all small letter symbols stand for non-negative integers.

In [5] I discussed pairs $\{a_n\}$, $\{k_n\}$ which represent the integers according to

Definition 1. $\{a_n\}$, $\{k_n\}$ represent the integers if, for each positive integer N there is one and only one vector $I = (i_1, i_2, \cdots, i_d)$ in V such that $N = a(I)$ and

$$i_{\nu+1} - i_\nu \geq k_\nu \quad \text{for} \quad 1 \leq \nu < d . \tag{4}$$

Let us write h and k for k_1 and k_2, respectively. Then it turns out ([5], theorems C and D) that $\{a_n\}$, $\{k_n\}$ represent the integers if and only if

$$0 \leq k - 1 \leq h \leq k = k_\nu \quad \text{for} \quad \nu \geq 2 , \tag{5}$$

and $\{a_n\}$ is the $(h,k)^{th}$ Fibonacci sequence $\{v_n\}$ defined by

$$\begin{cases} v_n = n & \text{for } 1 \leq n \leq k , \\ v_n = v_{n-1} + v_{n-h} & \text{for } k < n < h + k, \\ v_n = k - h + v_{n-1} + v_{n-k} & \text{for } n \geq h + k . \end{cases} \tag{6}$$

The Fibonacci sequence $\{u_n\}$ has been defined by authors in various ways, such as

$$\cdots, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 5, 8, 13, \cdots,$$
$$\cdots, 0, 0, 0, 0, 0, 0, 0, 1, 1, 2, 3, 5, 8, 13, \cdots,$$

and

$$\cdots, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, \cdots.$$

One can sometimes simplify an argument by changing from one definition to another. We chose to define $\{u_n\}$ by (2) in order to use (3). Sometimes it is more convenient to define (v_n) by

$$\begin{aligned} v_n &= 0 \quad \text{for } n < k^\star, \\ v_n &= 1 \quad \text{for } k^\star \le n \le 1, \qquad \text{and} \\ v_n &= k - h + v_{n-1} + v_{n-k} \quad \text{for } n \ge 2, \end{aligned} \tag{6.1}$$

where $k^\star = 1$ if $h = k - 1$ but $k^\star = -k + 2$ if $h = k$.

In the sequel, when we define a sequence, we will only consider the argument on hand at the time.

Next observe that the $(2,2)^{\text{th}}$ Fibonacci sequence is the ordinary Fibonacci sequence $\{u_n\}$, $n \ge 1$, and if $k_\nu = 2$ for all ν then condition (4) becomes condition (1). Thus in [5] I generalized Zeckendorf's theorem by replacing the constant 2 in (1) by a sequence $\{k_n\}$. Later, I replaced $\{k_n\}$ by an infinite matrix $M = [m_{\mu,\nu}]$, where $\mu, \nu \ge 1$, of non-negative integers $m_{\mu,\nu}$ as described in definition 2.

Definition 2. $\{a_n\}, M$ represent the integers if, for each positive integer N there is one and only one vector $I \in V$ such that $N = a(I)$ and

$$i_\mu - i_\nu \ge m_{\mu-\nu,\nu} \quad \text{for } 1 \le \nu < \mu \le d. \tag{7}$$

I described all such pairs $\{a_n\}, M$ to a splinter group of the 1962 International Congress of Mathematicians in Stockholm (see the programme). However in an effort to simplify my proofs, I made one further generalization as follows.

Definition 3. $\{a_n\}$, W represent the integers, where $W \subseteq V$, if for each positive integer N there is one and only one vector $I \in W$ such that $N = a(I)$.

There is very little one can say about $\{a_n\}$, W as this definition stands, so with my eye on condition (7), I make W satisfy axiom 2.

Axiom 2. If

$$(i_1, i_2, \cdots, i_d) \in V; \quad (j_1, j_2, \cdots, j_e) \in W; \quad 1 \le d \le e$$

and

$$i_{\nu+1} - i_\nu \ge j_{\nu+1} - j_\nu$$

for $1 \le \nu < d$ then

$$(i_1, i_2, \cdots, i_d) \in W .$$

This axiom merely says that if a vector is in W and we "cut its tail off" or "stretch" it, or do both things, it will still be in W. Important trivial consequences of axiom 2 are the laws

$$(8) \quad \begin{cases} (i_1, i_2, \cdots, i_d) \in W \iff (i_1 + 1, i_2 + 1, \cdots, i_d + 1) \in W \text{ and } i_1 \ge 1, \\ \text{and} \quad (i_1, i_2, \cdots, i_{d-1}, i_d) \in W \implies (i_1, i_2, \cdots, i_{d-1}, i_d + 1) \in W . \end{cases}$$

If $M = [m_{\mu,\nu}]$ is any matrix, and W is the set of all vectors $I = (i_1, i_2, \cdots, i_d)$ satisfying (7), then clearly axiom 2 holds for W. Hence definition 3 with axiom 2 is more general than definition 2, which in turn is more general than definition 1.

I will now state the fundamental theorem of all this work.

Theorem 1. Suppose $\{a_n\}$, W represent the integers, $W \subseteq V$, and axioms 1 and 2 hold. Then for $t = 1, 2, 3, \cdots$ all the integers N such that $a_t \le N < a_{t+1}$, and only these integers, each have a representation $N = a(I)$ with $I = (i_1, i_2, \cdots, i_d)$ in W and $i_d = t$.

It follows from the theorem that any part of a representation is a representation. In other words, if

$$(i_1, i_2, \cdots, i_d) \in W; \quad i \le e \le d$$

and

$$1 \le \nu_1 < \nu_2 < \cdots < \nu_e \le d$$

then

$$(i_{\nu_1}, i_{\nu_2}, \cdots, i_{\nu_e}) \in W .$$

Also the theorem shows that the representations of the successive integers $1, 2, 3, \cdots$ change "continuously," in the same way as their representations in the binary scale do. All possible representations using only $a_1, a_2, \cdots, a_t$ are exhausted before a_{t+1} is used. To determine the representation of a given integer N you find the suffix t such that $a_t \le N < a_{t+1}$, then the suffix s such that $a_s \le N - a_t < a_{s+1}$, and so on.

Now suppose W satisfies axiom 2. Then clearly $(1) \in W$, there is a least integer p such that $(1,p) \in W$, and there is a largest integer q such that the vector $(1, 1, \cdots, 1)$ of dimension q is in W. One of the numbers p and q is 1 and the other is greater than 1. My proofs, of theorem 1 and the results below for representations under definition 2, all split into the two cases $p = 1$ and $q = 1$. I always establish a chain of lemmas, each of which involves a number of complicated statements, and has a proof depending on nested induction arguments. One can gain some idea of the lengths of the proofs by inspecting [5]. For this reason, I do not intend to publish any proofs in this paper. I have tried repeatedly, but unsuccessfully, to find analytic proofs. I think that such proofs would be elegant, and would at the same time settle my monotonicity conjecture below.

An important result contained in the lemmas is the following:

If N is an integer $N \ge 1$, and the representations of N and $N + 1$ are respectively

$$N = a(i_1, i_2, \cdots, i_d)$$

and

$$N + 1 = (j_1, j_2, \cdots, j_e)$$

then

$$1 \le a(j_1 + 1,\ j_2 + 1, \cdots, j_e + 1) - a(i_1 + 1, i_2 + 1, \cdots, i_d + 1) \le q + 1\ .$$

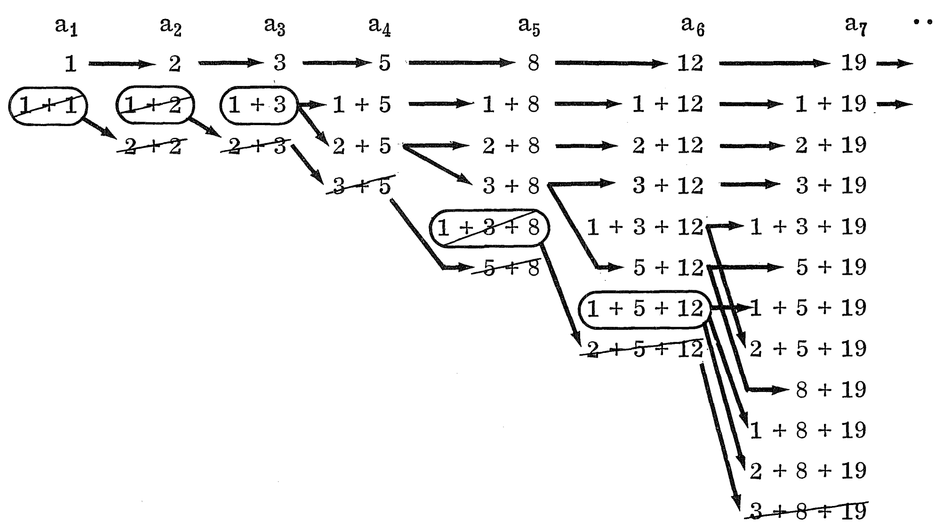
Notice the revelence of (8) to this result. Moreover the result enables us to give bounds for the rate of growth of $\{a_n\}$, and these bounds are necessary in the proofs. Taking $N = a_t - 1$ so that $N + 1 = a_t = a(t)$, we find that

$$1 \le a_{t+1} - a(i_1 + 1, i_2 + 1, \cdots, i_d + 1) \le q + 1.$$

We can in fact say more than the above, and I will illustrate the account by starting to construct a pair $\{a_n\}$, W inductively. We must have $a_1 = 1$, and the vector (1) in W. We are free to have (1,1) in W or not. Suppose we choose not to have it in. Then we can choose to have (1,2) in W or not. Suppose not. Then we are free to have (1,3) in W or not. Suppose we have it in. Then our construction could proceed as shown in Table 1.

Table 1

Construction of $\{a_n\}$, W when p = 3

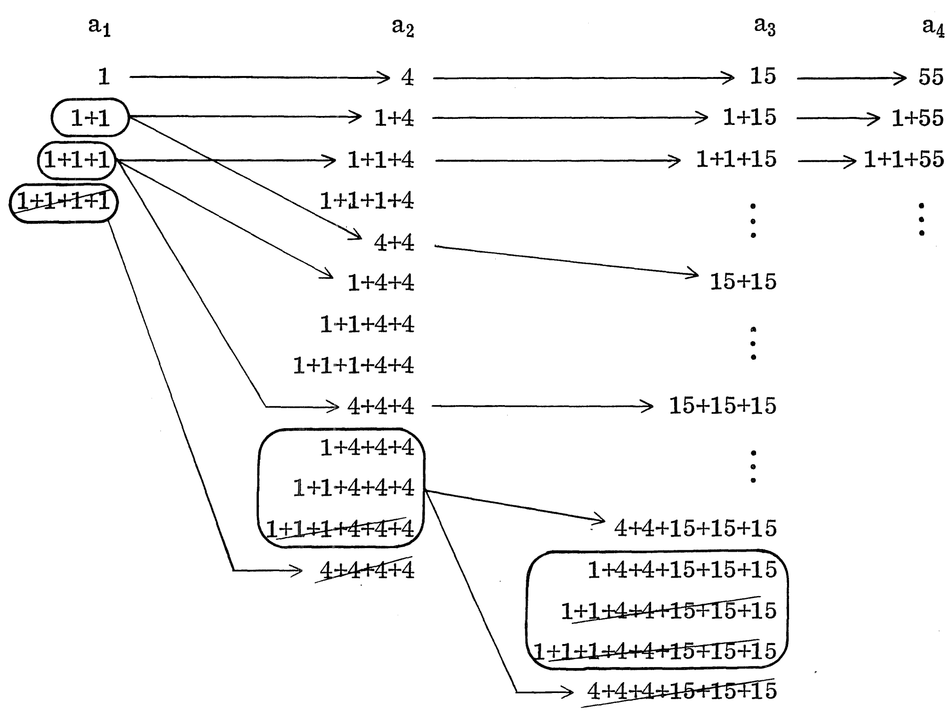


In the table, a representation is circled if at the appropriate stage of the construction, we had freedom to admit or reject it. A representation is crossed out iff it is not admitted. A representation at the head of an arrow must be admitted, or not as the case may be, by virtue of (8) or axiom 2, because the representation at the tail of the arrow was admitted or not. Notice that we had no freedom over the values of a_5 or a_8. Also the representation 1+3+12 must be admitted even though it is not controlled by (8) and earlier representations. If 1+3+12 is rejected, then $a_7 = 16$ and we have $17 = a(1,7) = a(4,6)$ contradicting the uniqueness of the representations. In general, for $p > 1$, when we have freedom over the value of a_t, (i.e., we can accept or reject some representation $N = a(i_1, i_2, \cdots, i_d)$ with $i_d = t - 1$), if we choose the lower value for a_t we will have freedom of choice over a_{t+1}. On the other hand, if we choose the higher value for a_t we will have no freedom over a_{t+1}, a_{t+2}, $\cdots, a_{t+p-2}$, and possibly over more terms, and sometimes over all further terms.

A typical construction with $q > 1$ is shown in Table 2.

Table 2

Construction of $\{a_n\}$, W when $q = 3$



Whichever way the pair $\{a_n\}$, W arise there will be a sequence $\{m_n\}$ of integers, $0 \le m_1 \le m_2 \le m_3 \le \cdots$, which may be finite or infinite, such that if we put

$$a_n = 0 \quad \text{for} \quad n \le 0, \tag{3}$$

then we have the identity

$$a_{n+1} = 1 + a_n + a_{n-m_1} + a_{n-m_2} + \cdots \qquad \text{for} \quad n \ge 0 \ . \tag{10}$$

This identity corresponds to (3). Moreover, if our use of the freedom of choice discussed above has a cyclic pattern, then $\{m_n\}$ is eventually periodic. It will then follow by subtracting equations (10) in pairs that high up terms in $\{a_n\}$ satisfy a finite recurrence relation. For example, continuing the construction of Table 1, let us use our freedom in column $3,4,5,\cdots$ according to the pattern: admit, no choice, reject, admit, no choice, reject,$\cdots$. Then $\{m_n\} = 2,5,8,11,14,\cdots$, an arithmetical progression with common difference 3, and $\{a_n\}$ is given by

$$\begin{cases} a_n = 0 \ \text{ for } \ n \le 0, \ a_1 = 1, \ a_2 = 2, \ a_3 = 3, \ \text{ and} \\ a_{n+1} = a_n + a_{n-2} + a_{n-2} - a_{n-3} \quad \text{for} \quad n \ge 3 \ . \end{cases} \tag{11}$$

The first 8 terms of $\{a_n\}$ are 1, 2, 3, 5, 8, 12, 19, 30 and the next 7 appear in (11.1) below.

We can use the above facts to obtain bounds for any sequence $\{a_n\}$ as follows. We define a sequence $\{b_n\}$ which has the same construction as $\{a_n\}$ to some particular stage, then from that stage on, whenever freedom arises we choose the largest (smallest) possible value for b_t. The sequence $\{b_n\}$ so constructed will satisfy a finite recurrence relation which we can use to evaluate the terms of $\{b_n\}$, and hence obtain upper (lower) bounds for $\{a_n\}$. As an example, let us find bounds for the sequence $\{a_n\}$ started in Table 1. If we admit as many representations as possible in the remainder of the construction, we find that $\{m_n\} = 2,5,7,10,12,15,\cdots$ (first differences $m_{n+1} - m_n$ are $3,2,3,2,3,2,\cdots$) and that $\{a_n\}$ may be defined as

$$(12)\qquad \begin{cases} a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = 5,\ a_5 = 8, \text{ and} \\ a_{n+1} = a_n + a_{n-2} + a_{n-4} \quad \text{for} \quad n \ge 5, \end{cases}$$

This sequence is the most highly divergent one which starts like Table 1.

Again starting from Table 1, we this time reject as many representations as possible. Then $\{m_n\}$ is the finite sequence $2, 5$ and

$$(13)\qquad \begin{cases} a_n = 0 & \text{for } n < 0 \\ a_{n+1} = 1 + a_n + a_{n-2} + a_{n-5} & \text{for } n \ge 0. \end{cases}$$

This sequence is the most slowly divergent one which starts like Table 1. The first 8 terms of any sequence starting like Table 1 are $1, 2, 3, 5, 8, 12, 19, 30$. I will now show some of the terms which follow these for the bounds (12) and (13), and for the example (11).

	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	$\cdots$
(12.1)	47	74	116	182	286	449	705	1107	$\cdots$
(11.1)	46	72	113	175	273	427	664	1035	$\cdots$
(13.1)	46	71	110	169	260	401	617	949	$\cdots$

Let us now consider the matrix M of definition 2. If there are three fixed integers r, s, t such that

$$m_{r,s} \le m_{r-t,s+t} + m_{t,s} \quad \text{with} \quad 1 \le t < r \quad \text{and} \quad 1 \le s,$$

then we will say that the element $m_{r,s}$ of M is redundant. We do so because if in some representation $N = a(I)$ we have

$$i_{r+s} - i_{s+t} \ge m_{r-t,s+t} \text{ and } i_{s+t} - i_s \ge m_{t,s}$$

then automatically

$$i_{r+s} - i_s \ge m_{r,s}\,.$$

There is in fact no loss of generality in assuming that redundant elements take the largest possible value (which does not alter the representations under definition 2). In other words (applying an extension of the above argument) we assume that

$$m_{r,s} \geq m_{r-t,s+t} + m_{t,s} \quad \text{for all } 1 \leq t < r \text{ and } 1 \leq s .$$

We extend the definition of redundancy to rows, by saying that a row of M is redundant if every element of the row is redundant. If any one element of a row is not redundant then we say that the row is non-redundant.

Next let us assume that $\{a_n\}$, M represent the integers. Then it turns out that the matrix M has only two kinds of row, namely "straight" rows like

$$(\alpha, \alpha, \alpha, \alpha, \cdots), \quad 0 \leq \alpha$$

and "bent" rows of the form

$$(\beta, \alpha, \alpha, \alpha, \cdots) \text{ where } 0 \leq \beta = \alpha - 1.$$

If either type of row is non-redundant then every element α in it is non-redundant. If a bent row is non-redundant then every succeeding row is redundant (the bent row is the last non-redundant row of M). Moreover, if a bent row is non-redundant, and its element β is non-redundant, then it is the first non-redundant row of M, and if in addition $\beta > 0$ then it is the very first row of M. It follows from these facts that if M has infinitely many non-redundant rows, then all its rows are straight.

If the row

$$(m_{r,1}, m_{r,2}, m_{r,3}, \cdots)$$

of M is non-redundant, then

$$m_r^\star \leq m_{r,1} \leq 1 + m_r^\star ,$$

where

$$m_r^\star = \underset{1 \le t < r}{\text{maximum}} \left| m_{r-t,1+t} + m_{t,1} \right| .$$

This condition merely says that either $m_{r,1}$ is redundant or it lays the weakest possible new condition on the representations. Now we already know that

$$m_{r,1} \le m_{r,2} \le m_{r,1} + 1 .$$

Hence it follows that (even if $m_{r,1}$ is redundant), either $m_{r,2}$ imposes the same condition as $m_{r,1}$, or $m_{r,2}$ imposes the weakest condition on the representations, which is stronger than that imposed by $m_{r,1}$.

Satisfying the above rules in all possible ways produces all possible matrices M for which there is a sequence $\{a_n\}$ such that $\{a_n\}$, M represent the integers. For example, the first corner of any matrix which starts with $m_{11} = 2$ looks like one of the matrices in Table 3.

Table 3

$$\begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 4 & 4 & \cdots \\ 6 & 6 & 6 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 4 & 4 & \cdots \\ 6 & 7 & 7 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 4 & 4 & \cdots \\ 7 & 7 & 7 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 4 & 4 & \cdots \\ 7 & 8 & 8 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 & \cdots \\ 4 & 5 & 5 & \cdots \\ 6 & 7 & 7 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & \cdots \\ 7 & 7 & 7 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & \cdots \\ 7 & 8 & 8 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & \cdots \\ 8 & 8 & 8 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & \cdots \\ 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix} \quad \begin{bmatrix} 2 & 3 & 3 & \cdots \\ 5 & 6 & 6 & \cdots \\ 8 & 9 & 9 & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

Once the matrix $M = \left[m_{\mu,\nu}\right]$ is given, the sequence $\{a_n\}$ is determined by (9) and (10), provided $\{m_\mu\}$ is derived from M as follows. If M has no bent row then $\{m_\mu\}$ is infinite and

$$m_\mu = m_{\mu,1} \quad \text{for} \quad 1 \le \mu .$$

On the other hand if M has a first bent row, and this row is the ρ^{th} row, then $\{m_\mu\}$ is finite with ρ terms given by

$$m_\mu = m_{\mu,1} \quad \text{for } 1 \le \mu \le \rho .$$

We get a simplification of (9) and (10) in the case when M has no bent row, but it has only a finite number of non-redundant rows. In this case, if the last non-redundant row is the ρ^{th}, then $\{m_\mu\}$ is periodic with period $m_\rho = m_{\rho,1}$. Hence not only (9) and (10) hold, but we also find by subtraction that

$$(14) \qquad a_{n+1} = a_n + a_{n-m_1} + a_{n-m_2} + \cdots + a_{n-m_{\rho-1}} + a_{n-m_\rho+1} \quad \text{for } n \ge m_\rho .$$

It is easy to see how relations (9), (10), and (14) generalize the definition (6) of the $(h,k)^{th}$ Fibonacci sequence.

When we know that all rows of M after the ρ^{th} row are redundant, we usually remove them from M. Then M has order $\rho \times \infty$ instead of $\infty \times \infty$. However, the fact that M has order $\rho \times \infty$ does not necessarily imply that the ρ^{th} row is non-redundant.

The bounding sequences (12) and (13) which we found earlier are in fact the sequences $\{a_n\}$ for the matrices

$$M = \begin{bmatrix} 2 & 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & 5 & \cdots \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} 2 & 2 & 2 & 2 & \cdots \\ 5 & 6 & 6 & 6 & \cdots \end{bmatrix},$$

respectively. In these cases, our constructive process of "admitting (rejecting) as many representations as possible" is equivalent to saying, "let all rows of the matrix after the 2nd be redundant." The sequence (11) corresponds to the $\infty \times \infty$ matrix

$$M = \begin{bmatrix} 2 & 2 & 2 & 2 & \cdots \\ 5 & 5 & 5 & 5 & \cdots \\ 8 & 8 & 8 & 8 & \cdots \\ 11 & 11 & 11 & 11 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} .$$

With this matrix, a vector $(i_1, i_2, \cdots, i_d)$ satisfies condition (7) iff (i) we have $i_{\nu+1} - i_\nu \geq 2$ for $1 \leq \nu < d$, and (ii) if $1 \leq \eta < \theta < d$ and

$$i_{\eta+1} - i_\eta = i_{\theta+1} - i_\theta = 2$$

then there is an integer λ such that $\eta < \lambda < \theta$ and $i_{\lambda+1} - i_\lambda \geq 4$.

It has long been known that the Fibonacci sequence $\{u_n\}$ can be obtained from Pascal's triangle. The triangle is set out on the lattice points of the first quadrant of the (x, y)-plane. Then one draws a family of equispaced parallel lines on the triangle, choosing the slope and spacing of the lines, so that the sum of all the numbers of the triangle, whose lattice points lie on the n^{th} line of the family, is the n^{th} term u_n of the sequence. In 1959, I observed that the $(h,k)^{th}$ Fibonacci sequence $\{v_n\}$ could be obtained in the same way, provided that when $h = k - 1$ the first row $(1, 1, 1, \cdots)$ of the triangle is removed from the triangle ([6] theorem 8).

Harris and Styles defined sequences by means of Pascal's triangle in [9], and discussed the properties of their sequences. Suppose Pascal's triangle lies on the lattice points of the first quadrant of the (x, y)-plane. Then for $p \geq 0$, $q > 0$ they let $u(n, p, q)$ be the sum of the n^{th} term in the first row $(1, 1, 1, \cdots)$ of the triangle and those terms of the triangle which can be reached from it by taking steps $(x, y) \to (x - p - q, y + q)$. When $q = 1$, we have

$$u(n, p, 1) = v_{n-p+1} \quad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots$$

where $\{v_n\}$ is the $(p+1, p+1)^{th}$ Fibonacci sequence (6.1).

Now suppose that M, $\{a_n\}$ represent the integers, and that all rows of M after the ρ^{th} are redundant. Then the terms of $\{a_n\}$ can be obtained from a $\rho + 1$ dimensional Pascal's triangle. The n^{th} term of $\{a_n\}$ is the sum of all the numbers of the generalized triangle which lie on the n^{th} number of a ρ-dimensional family of equispaced parallel hyperplanes. I will give the details for $\rho = 2$ and the second row non-redundant. The reader will immediately see the result for general ρ. With slight modifications, the method can be applied to a wide class of sequences satisfying finite recurrence relations.

When $\rho = 2$ and the second row is non-redundant, the matrix M is of the form

$$M = \begin{bmatrix} \alpha & \alpha & \alpha & \alpha & \cdots \\ \beta & \gamma & \gamma & \gamma & \cdots \end{bmatrix}$$

where

$$0 \le 2\alpha \le \beta \le 2\alpha + 1 \le \gamma \quad \text{and} \qquad \beta \le \gamma \le \beta + 1 .$$

The second row of M could be either straight or bent. Also the sequence $\{a_n\}$ is given by

$$\text{(15)} \qquad \begin{cases} a_n = 0 & \text{for} \quad n < \alpha^\star , \\ a_n = 1 & \text{for} \quad \alpha^\star \le n \le 1 , \\ a_{n+1} = \gamma - \beta + a_n + a_{n-\alpha} + a_{n-\gamma+1} & \text{for} \quad n \ge 1, \end{cases}$$

where $\alpha^\star = 1$ if $\beta = \gamma - 1$ but $\alpha^\star = -\alpha + 1$ if $\beta = \gamma$.

Notice that when $\alpha = 2$ and $\beta = \gamma = 5$ then we get back to the sequence (12) again.

We now define our 3-dimensional Pascal's triangle. In other words, we define an integer-valued function $\pi(x,y,z)$ on the 3-dimensional lattice by the relations

$$\pi(x,y,z) = \begin{cases} 0 \text{ if } x < 0 \text{ or } y < 0 \text{ or } z < 0, \\ 1 \text{ if } x = y = z = 0, \\ \pi(x-1,y,z) + \pi(x,y-1,z) + \pi(x,y,z-1) \text{ otherwise} \end{cases}$$

It is easy to see that Pascal's triangle appears on each of the three planes $x = 0$, $y = 0$ and $z = 0$. My result is that the n^{th} term of $\{a_n\}$ of (15) is the sum of all the values of $\pi(x,y,z)$ (whose lattice points lie) on the plane

$$x + (\alpha + 1)y + \gamma z = n + \alpha - 1 + (\gamma - \beta)\alpha$$

provided that if $\gamma = \beta - 1$ we remove the x-axis (i.e., if $\gamma = \beta - 1$, we replace π by $\pi^\star$ where $\pi^\star(x,y,z) = 0$ if $y = z = 0$ but $\pi^\star = \pi$ otherwise). The proof is by induction.

Next let r be a fixed integer $r \ge 1$. Let $(i_1, i_2, \cdots, i_d) \in W$ iff

$$i_{\nu+r} - i_\nu \geq 1$$

for $1 \leq \nu \leq d - r$, and put $b_n = (r+1)^{n-1}$ for $n \geq 1$. Then $\{b_n\}$, W represent the integers in the familiar scale of powers of $r + 1$, and the order of the terms in $\{b_n\}$ is immaterial. Suppose, on the other hand, that

$$(i_1, i_2, \cdots, i_d) \in W$$

iff

$$i_{\nu+1} - i_\nu \geq 2$$

for

$$1 \leq \nu < d,$$

and $\{b_n\}$, W represent the integers. Then as I showed in [1] axiom 1 must hold, and in fact $\{b_n\} = \{u_n\}$.

I would now like to state my monotonicity conjecture, which extends a conjecture that I made in [5].

Conjecture. Suppose $\{b_n\}$, W represent the integers and axiom 2 holds. Then either axiom 1 holds for $\{b_n\}$ or $\{b_n\}$ is $s^0, s^1, s^2, \cdots$ in some order and s is an integer $s \geq 1$.

Another result which gives weight to the conjecture is

Theorem 2. Let $r \geq 1$ be a fixed integer. Let M be the matrix whose only non-redundant row is its r^{th} row, and this r^{th} row is $(0, 1, 1, \cdots)$. If $\{b_n\}$, M represent the integers then axiom 1 holds for $\{b_n\}$. Moreover, $b_1 = 1$ and $b_{n+1} = (r+1)b_n + 1$ for $n \geq 1$.

The first example of a pair $\{a_n\}$, W which is not equivalent to a pair $\{a_n\}$, M was given by my student A. J. W. Hilton in 1963. He took a fixed integer $r \geq 4$, and let W be the set of all vectors I of V such that

$$1 \leq i_1 < i_2 < \cdots < i_d$$

and, for $1 < \nu < d$, if $i_{\nu+1} - i_\nu = 1$ then $i_\nu - i_{\nu-1} \geq r$. Then he put

$$w_n = 1 \quad \text{for} \quad n \le 1$$
$$w_2 = 2$$
$$w_n = w_{n-1} + w_{n-2} + w_{n-p} \quad \text{for} \quad n \ge 3,$$

With these definitions, we have

Theorem 3 (Hilton). $\{w_n\}$, W represent the integers but are not equivalent to any system $\{b_n\}$, M.

I have tried to find an elegant classification for all sets $\{a_n\}$, W. However, I have so far been unable to improve on the constructive method which I have described for obtaining all sets $\{a_n\}$, W.

In this paper, I have been concerned with unique representations. It would be interesting to know what happens if the uniqueness condition was dropped, and perhaps only sufficiently large numbers N had to have a representation (N > constant). Results in this direction have been found by Brown, Ferns, Hoggatt, King, and others [1], [2], [3], [7], and [8], respectively. I feel that the results I have given in this paper are complete in the same sense as N. G. de Bruijn's discussion is complete for representations

$$N = s_1 + s_2 + s_3 + \cdots ,$$

where each s_i belongs to a finite or infinite set S_i of non-negative integers containing 0. In a paper [4] which is now classical, he showed that all such systems are what he called "degenerate British number systems."

Some results have been obtained concerning representing all integers in some interval by Harris, Hilton, Hoggatt, Mohanty, Styles, myself and others [6], [9]. However, the problems concerning representations for all the integers $0, \pm 1, \pm 2, \cdots$ are much more difficult and only a few special theorems are known.

REFERENCES

1. J. L. Brown, Jr., "Note on Complete Sequences of Integers," Amer. Math. Monthly, Vol. 68 (1961), pp. 557-560.
2. J. L. Brown, Jr., "Generalized Bases for the Integers," Amer. Math. Monthly, Vol. 71 (1964), pp. 973-980.

3. J. L. Brown, Jr., "A New Characterization of the Fibonacci Numbers," The Fibonacci Quarterly, Vol. 3 (1965), pp. 1-8.
4. N. G. de Bruijn, "On Number Systems," Nieuw. Archief voor Wiskunde, Vol. 3 (1956), pp. 15-17.
5. D. E. Daykin, "Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers," Jour. London Math., Vol. 35 (1960), pp. 143-161.
6. D. E. Daykin and A. J. W. Hilton, "Bases for Infinite Intervals of Integers," The Fibonacci Quarterly, Vol. 5, No. 4 (1967), pp. 329-346.
7. H. H. Ferns, "On the Representation of Integers as Sums of Distinct Fibonacci Numbers," The Fibonacci Quarterly, Vol. 3 (1965), pp. 21-30.
8. R. L. Graham, "A Property of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2 (1964), pp. 1-10.
9. V. C. Harris and C. C. Styles, "A Generalization of Fibonacci Numbers," The Fibonacci Quarterly, Vol. 2 (1964), pp. 277-289.

★ ★ ★ ★ ★

[Continued from page 493.]

3. G. Gompertz, 'On the Nature of the Function Expressive of the Law of Human Mortality, and on a New Mode of Determining the Value of Life Contingencies," Phil. Trans. Roy. Soc., London, 115 (1825), 513-585.
4. J. P. Gram, Om Räkkeudviklinger Bestemte ved Hjälp af de mindste Kvadraters Methode, Kjöpenhavn (1879).
5. C. Jordan, Calculus of Finite Differences, Chelsea Pub. Co., New York, 1950.
6. T. J. Steiljes, "Tables des valeurs des Sommes $S_k = \sum_{n=1}^{\infty} n^{-k}$," Acta Mathematica, 10 (1887), 299-302.
7. G. Szegö, "Orthogonal Polynomials," Amer. Math. Soc. Colloquium Publications, 23 (1959).

★ ★ ★ ★