RECURRENCE FORMULAS

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In this paper p(n) shall denote, as usual, the number of partitions of n; that is, the number of solutions of the equation:

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n$$

in non-negative integers. We state the following identity

$$p(n) = -\sum_{\substack{0 \le i \le m \\ m < j \le n}} p(i)\, e(j - i)\, p(n - j) \ , \tag{1}$$

where $e(k) = (-1)^k$ if $k = \frac{1}{2}(3h^2 \pm h)$, 0 otherwise, and $p(0) = 1$.

The proof of (1) will be evident as a special case of the following more general form. (See acknowledgement.) Put

$$f(x) = \sum_{n=0}^{\infty} a(n)\, x^n, \quad (f(x))^{-1} = \sum_{n=0}^{\infty} b(n) x^n \ ,$$

where for convenience $a(0) = b(0) = 1$. Then

$$\sum_{j=0}^{n} a(j)\, b(n - j) = 0 \qquad (n > 0) \ . \tag{2}$$

Now consider the sums

$$S = \sum_{\substack{0 \le i \le m \\ m < j \le n}} n(i)\, b(j - i)\, a(n - j) \ ,$$

$$T = \sum_{0 \le i \le j \le m} a(i) b(j - i)\, a(n - j)$$

where $0 < m < n$. Then in the first place, by (2),

$$T = \sum_{j=0}^{m} a(n - j) \sum_{i=0}^{j} a(i)\, b(j - i) = a(n) . \tag{3}$$

In the next place,

$$S + T = \sum_{0 \le i \le m} \sum_{i \le j \le n} a(i)\, b(j - i)\, a(n - j)$$

$$= \sum_{0 \le i \le m} a(i) \sum_{s=0}^{n-i} b(s)\, a(n - i - s) .$$

The inner sum on the extreme right vanishes unless $n - i = 0$; since $m < n$ this condition is satisfied for no value of i in the range $0 \le i \le m$ and therefore $S + T = 0$.

Combining this with (3), we get $S = -a(n)$, or, explicitly,

$$\sum_{\substack{0 \le i \le m \\ m < j \le n}} a(i)\, b(j - i)\, a(n - j) = -a(n) \qquad (0 < m < n) . \tag{4}$$

The recurrence (1) clearly follows from (4).

Note. Since we may equally well have started out with $(f(x))^{-1}$ rather than $f(x)$, we have also

$$\sum_{\substack{0 \le i \le m \\ m < j \le n}} b(i)\, a(j - i)\, b(n - j) = -b(n) \qquad (0 < m < n) .$$

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