ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico, 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-178 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida.

For all positive integers n show that

$$F_{2n+2} = \sum_{i=1}^{n} 2^{n-i} F_{2i-1} + 2^n$$

and

$$F_{2n+3} = \sum_{i=1}^{n} 2^{n-i} F_{2i} + 2^{n+1} \quad .$$

Generalize.

B-179 Based on Douglas Lind's Problem B-165.

Let Z^+ consist of the positive integers and let the function b from Z^+ to Z^+ be defined by $b(1) = b(2) = 1$, $b(2k) = b(k)$, and $b(2k+1) = b(k+1) + b(k)$ for $k = 1, 2, \cdots$. Show that every positive integer m is a value of b(n) and that $b(n+1) \geq b(n)$ for all positive integers n.

B-180 Proposed by Reuben C. Drake, North Carolina A T University, Greensboro, North Carolina.

Enumerate the paths in the Cartesian plane from $(0,0)$ to $(n,0)$ that consist of directed line segments of the four following types:

Type	I	II	III	IV
Initial Point	$(k,0)$	$(k,0)$	$(k,1)$	$(k,1)$
Terminal Point	$(k,1)$	$(k+1,0)$	$(k+1,1)$	$(k+1,0)$

B-181 Proposed by J. B. Roberts, Reed College, Portland, Oregon.

Let m be a fixed integer and let $G_{-1} = 0$, $G_{\delta} = 1$, $G_n = G_{n-1} + G_{n-2}$ for $n \geq 1$. Show that $G_0, G_m, G_{2m}, G_{3m}, \cdots$ is the sequence of upper left principal minors of the infinite matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ G_{m-2} & G_{m-2}+G_m & 1 & 0 & \cdots \\ 0 & (-1)^m & G_{m-2}+G_m & 1 & \cdots \\ 0 & 0 & (-1)^m & G_{m-2}+G_m & \cdots \\ 0 & 0 & 0 & (-1)^m & \cdots \\ & \vdots & & & \end{pmatrix}$$

B-182 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida.

Show that for any prime p and any integer n,

$$F_{np} \equiv F_n F_p \pmod{p} \quad \text{and} \quad L_{np} \equiv L_n L_p \equiv L_n \pmod{p}.$$

B-183 Proposed by Gustavus J. Simmons, Sandia Corporation, Albuquerque, New Mexico.

A positive integer is a palindrome if its digits read the same forward or backward. The least positive integer n such that n^2 is a palindrome but n is not is 26. Let S be the set of n such that n^2 is a palindrome but n is not. Is S empty, finite, or infinite?

SOLUTIONS

FIBONACCI PYTHAGOREAN TRIPLES

B-160 Proposed by Robert H. Anglin, Dan River Mills, Danville, Virginia.

Show that if $x = F_nF_{n+3}$, $y = 2F_{n+1}F_{n+2}$, and $z = F_{2n+3}$, then

$$x^2 + y^2 = z^2 .$$

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

Let $u = F_{n+2}$ and $v = F_{n+1}$. Then

$$u^2 - v^2 = (u + v)(u - v) = F_{n+3}F_n = x, \qquad 2uv = y, \qquad u^2 + v^2 = z,$$

and hence $x^2 + y^2 = z^2$.

Also solved by Herta T. Freitag, Bruce W. King, Douglas Lind, John W. Milsom, A. G. Shannon (Boroko, T. P. N. G.), Gregory Wulcyzn, and the Proposer.

PELL NUMBER IDENTITIES

B-161 Proposed by John Ivie, Student, University of California, Berkeley, California.

Given the Pell numbers defined by $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$, show that for $k > 0$:

(i) $$P_k = \sum_{r=0}^{[(k-1)/2]} \binom{k}{2r+1} 2^r .$$

(ii) $$P_{2k} = \sum_{r=1}^{k} \binom{k}{r} 2^r P_r .$$

Solution by Douglas Lind, Cambridge University, Cambridge, England.

Let

$$a = 1 + \sqrt{2}, \quad b = 1 - \sqrt{2}$$

be the roots of the characteristic polynomial $x^2 - 2x - 1$. It follows from the theory of difference equations that there are constants A and B such that

$$P_n = Aa^n + Bb^n .$$

Solving the system of simultaneous equations resulting by setting $n = 0, 1$, we find

$$A = 1/2\sqrt{2}, \qquad B = -1/2\sqrt{2} .$$

Hence

$$P_k = \frac{1}{2\sqrt{2}}(a^k - b^k) = \frac{1}{2\sqrt{2}} \sum_{j=0}^{k} \binom{k}{j}\left[2^{j/2} - (-1)^j 2^{j/2}\right]$$

$$= \frac{1}{2\sqrt{2}} \sum_{r=0}^{[\frac{1}{2}(k-1)]} \binom{k}{2r+1}\left[2 \cdot 2^{\frac{1}{2}(2r+1)}\right] = \sum_{r=0}^{[\frac{1}{2}(k-1)]} \binom{k}{2r+1} 2^r .$$

Also, since a and b satisfy $x^2 = 2x + 1$, we have

$$P_{2k} = \frac{1}{2\sqrt{2}}(a^{2k} - b^{2k}) = \frac{1}{2\sqrt{2}}\left((2a + 1)^k - (2b + 1)^b\right)$$

$$= \sum_{r=0}^{k} \binom{k}{r} 2^r \left(\frac{a^r - b^r}{2\sqrt{2}}\right) = \sum_{r=0}^{k} \binom{k}{r} 2^r P_r .$$

Also solved by Herta T. Freitag, Bruce W. King, Gregory Wulcyzn, Michael Yoder, and the Proposer.

A REPRESENTATION THEOREM

B-162 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let r be a fixed positive integer and let the sequence $u_1, u_2, \cdots$ satisfy $u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-r}$ for $n > r$ and have initial conditions $u_j = 2^{j-1}$ for $j = 1, 2, \cdots, r$. Show that every representation of U_n as a sum of distinct u_j must be of the form u_n itself or contain explicitly the terms $u_{n-1}, u_{n-2}, u_{n-r+1}$ and some representation of u_{n-r}.

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

For $r = 1$, the theorem is trivial; we therefore assume $r \geq 2$. First we show by induction that

$$\sum_{i=1}^{n} u_i < u_{n+2} .$$

For $n = 1, 2, \cdots, r$ this is obvious; and if

$$\sum_{i=1}^{n} u_i < u_{n+2}, \qquad \text{where} \qquad n < r ,$$

$$\sum_{i=1}^{n+1} u_i < u_{n+2} + u_{n+1} \leq u_{n+3} .$$

Now let

$$u_n = \sum_{k<n} c(k) u_k ,$$

$c(k) = 0$ or 1 for all k, be a representation of u_n and assume $c(j) = 0$ for some j with $n - r + 1 \leq j \leq n - 1$. Then

$$\begin{aligned} \sum_{k<n} c(k) u_k &< \sum_{k=1}^{n-1} u_k - u_j \\ &\leq \left(\sum_{k=1}^{n-r-1} u_k + \sum_{k=n-r}^{n-1} u_k \right) - u_{n-r+1} \\ &= \left(\sum_{k=1}^{n-r-1} u_k - u_{n-r+1} \right) + u_n < u_n , \end{aligned}$$

which is a contradiction. Thus we must have

$$u_n = u_{n-1} + \cdots + u_{n-r+1} + S ,$$

where S is some representation of u_{n-r}.

See "Generalized Fibonacci Numbers and the Polygonal Numbers," Journal of Recreational Mathematics, July, 1968, pp. 144-150.

Also solved by the Proposer.

A VARIANT OF THE EULER-BINET FORMULA

B-163 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let n be a positive integer. Clearly

$$(1 + \sqrt{5})^n = a_n + b_n\sqrt{5}$$

with a_n and b_n integers. Show that 2^{n-1} is a divisor of a_n and of b_n.

Solution by David Zeitlin, Minneapolis, Minnesota.

Let

$$\alpha = (1 + \sqrt{5})/2 \quad \text{and} \quad \beta = (1 - \sqrt{5})/2 .$$

Elimination of β^n from $L_n = \alpha^n + \beta^n$ and $\sqrt{5}F_n = \alpha^n - \beta^n$ gives

$$2\alpha^n = L_n + \sqrt{5}F_n .$$

Thus,

$$(1 + \sqrt{5})^n = 2^{n-1}L_n + \sqrt{5}(2^{n-1}F_n),$$

where $\alpha_n = 2^{n-1}L_n$ and $b_n = 2^{n-1}F_n$.

Also solved by Juliette Davenport, Herta T. Freitag, John E. Homer, Jr., John Ivie, Bruce W. King, Douglas Lind, Peter A. Lindstrom, A. G. Shannon (Boroko, T. P. N. G.), Michael Yoder, and the proposer.

A GENERALIZED SEQUENCE WITH CHARACTERISTIC 11,111

B-164 Proposed by J. A. H. Hunter, Toronto, Canada.

A Fibonacci-type sequence is defined by:

$$G_{n+2} = G_{n+1} + G_n ,$$

with $G_1 = a$ and $G_2 = b$. Find the minimum positive values of integers a and b, subject to a being odd, to satisfy:

$$G_{n-1}G_{n+1} - G_n^2 = -11111(-1)^n \quad \text{for} \quad n > 1.$$

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

If the given equation is true for any one value of n, it is true for all values of n; hence taking $n = 2$, we get

$$a(a + b) - b^2 = -11111 ,$$
$$4a^2 + 4ab - 4b^2 = -44444,$$
$$(2a + b)^2 = 5b^2 - 44444 .$$

Now $5b^2 - 44444 > b^2$ leads to $b > 105$; trying $b = 106, 107, \cdots$ in succession, one finds the smallest value of b to make $5b^2 - 44444$ a square $b = 111$. However, this gives $2a + b = 131$, $a = 10$, and a is supposed to be odd. Continuing with $b = 112, 113, \cdots$, we find

$$166^2 = 5(120)^2 - 44444 ,$$

which gives $a = 23$, $b = 120$ as the smallest solution.

Also solved by Christine Anderson, Herta T. Freitag, John E. Homer, Jr., Gregory Wulczyn, and the Proposer.

A MONOTONIC SURJECTION FROM Z^+ TO Z^+

B-165 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Define the sequence $\{b(n)\}$ by $b(1) = b(2) = 1$, $b(2k) = b(k)$, and

$$b(2k + 1) = b(k + 1) + b(k) \quad \text{for} \quad k \geq 1 .$$

For $n \geq 1$, show the following:

(a) $$b([2^{n+1} + (-1)^n /3) = F_{n+1} .$$

(b) $$b([7 \cdot 2^{n-1} + (-1)^n]/3) = L_n .$$

Solution by Michael Yoder, Student, Albuquerque Academy, Albuquerque, New Mexico.

(a) For $n = 0, 1$ the formula is easily verified. Assume it is true for $n - 2$ and $n - 1$ with $n \geq 2$; then if n is even,

$$\begin{aligned} b\left[(2^{n+1} + 1)/3\right] &= b\left[(2^n - 1)/3 + b\ (2^n + 2)/3\right] \\ &= F_n + b\left[(2^{n-1} + 1)/3\right] \\ &= F_n + F_{n-1} = F_{n+1} . \end{aligned}$$

Similarly, if n is odd,

$$b\left[(2^{n+1} - 1/3\right] = F_{n+1} .$$

(b) For $n = 1, 2$ the theorem is true; and by exactly the same argument as in (a), it follows by induction for all positive integers n.

Also solved by Herta T. Freitag and the Proposer.

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(Continued from page 101.)

SOLUTIONS TO PROBLEMS

1. $5n^3 - 4n^2 + 3n - 8$.
2. $3n^2 - 8n + 4$ and the Fibonacci sequence: $1, 4, 5, 9, 14, \cdots$.
3. $7n^3 + 3n^2 - 5n + 2 + 3x2^n$.
4. $4n + 3 + 3(-1)^n$.
5. $2n^3 - 3n^2 - n + 5$ and the Fibonacci sequence $4L_n$.

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