GENERALIZED FIBONACCI k-SEQUENCES

HYMAN GABAI
York College (CUNY) and University of Illinois (UICSM)

1. INTRODUCTION

For $k \geq 2$, the Fibonacci k-sequence $F(k)$ may be defined recursively by

$$f_n = 0 \quad (n \leq 0), \quad f_1 = 1, \quad f_n = \sum_{i=n-k}^{n-1} f_i \quad (n > 1).$$

A generalized Fibonacci k-sequence $A(k)$ may be constructed by arbitrarily choosing $a_1, a_0, a_{-1}, \cdots, a_{2-k}$, and defining

$$a_n = 0 \quad (n < 2 - k), \qquad a_n = \sum_{i=n-k}^{n-1} a_i \quad (n > 1).$$

In this paper, some well-known properties of $F(2)$ (see [1] and [8]) are generalized to the sequences $A(k)$. For some properties of $F(k)$, see [4], [6], and [7]. The sequences $A(3)$ are investigated in [9].

The pedagogical values of introducing Fibonacci sequences in the classroom are well known. (See, for example [3], pp. 336-367.) It seems possible that the generalizations described in this paper may suggest some areas of investigation suitable for high school and college students. (See, for example [5].) For once a theorem concerning $F(2)$ has been discovered, one may search for corresponding theorems concerning $A(2)$, $F(3)$, $A(3)$, $\cdots$ and finally $F(k)$ and $A(k)$. (See [2].)

2. THEOREMS

The first theorem is a "shift formula" needed in the proof of Theorem 6.

<u>Theorem 1.</u> For $n \geq 2$, $a_{n+1} = 2a_n - a_{n-k}$.

Theorem 2 is a generalization of the theorem that any two consecutive terms of F(2) are relatively prime.

Theorem 2. For $n \geq 2$, every common divisor of

$$a_n, \; a_{n+1}, \; a_{n+2}, \; \cdots, \; a_{n+k-1}$$

is a divisor of $a_2, a_3, \cdots, a_{n-1}$.

Some summation theorems are given in Theorems 3, 4, and 5.

Theorem 3. (a) For $n \geq 1$ and $m \geq 1$,

$$\sum_{i=0}^{n} a_{ki+m+1} = \sum_{i=m+1-k}^{kn+m} a_i \; .$$

(b) For $n \geq 1$,

$$\sum_{i=1}^{n} a_{ki} = \sum_{i=0}^{kn-1} a_i \; .$$

(c) For $n \geq 1$,

$$a_{kn} - a_0 = \sum_{\substack{1 \leq i \leq kn-1 \\ i \not\equiv 0 \pmod{k}}} a_i$$

Theorem 4. For $n \geq 2 - k$,

$$\sum_{i=2-k}^{n} a_i = \frac{1}{k-1}\left(a_{n+k} - a_1 + \sum_{i=1}^{k-2} i a_{-i} - \sum_{i=1}^{k-2} (k - i - 1) a_{n+i}\right) \; .$$

Theorem 5. For $n \geq 1$,

$$\sum_{i=1}^{n} a_i^2 = a_n a_{n+1} - a_1 a_0 - \sum_{j=2}^{k-1} \sum_{i=1}^{n} a_i a_{i-j} \; .$$

Theorems 6, 7, and 8 show relations between F(k) and A(k).

Theorem 6. For $n \geq 1$ and $m \geq 1$,

$$a_{n+m} = \sum_{j=1}^{k} \left(a_{n-k+j} \sum_{i=1}^{j} f_{m-j+1} \right) .$$

Theorem 7. Let d_m be the greatest common divisor of

$$f_m,\ f_{m-1},\ \cdots,\ f_{m-k+2} .$$

If $m \geq 1$, m divides n, and d_m divides a_m, then d_m divides a_n.

Theorem 8. Let r be the largest root of the polynomial equation

$$x^k - \sum_{i=0}^{k-1} x^i = 0 .$$

Then

(a) $$\lim_{n\to\infty}\left(\frac{a_n}{f_n}\right) = \frac{1}{r^k}\sum_{j=2}^{k+1}\left(a_{j-k}\sum_{i=1}^{j-1} r^{k-i}\right) ,$$

and

(b) $$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = r .$$

3. PROOFS OF THEOREMS

Theorem 1 follows directly from the definition of A(k). For, if $n \geq 2$, then

$$a_{n+1} = \sum_{i=n-k+1}^{n} a_i = \sum_{i=n-k}^{n-1} a_i + a_n - a_{n-k} = 2a_n - a_{n-k} .$$

To prove Theorem 2, suppose that d is a common divisor of a_n, a_{n+1}, $\cdots, a_{n+k-1}$. Since

$$a_{n+k-1} = \sum_{i=n-1}^{n+k-2} a_i = a_{n-1} + \sum_{i=n}^{n+k-2} a_i ,$$

it follows that d also divides a_{n-1}. It follows, by induction, that d divides $a_{n-2}, \cdots, a_2$.

For the proof of Theorem 3(a), choose any integer $m \geq 1$. Now Theorem 3(a) holds for $n = 1$ because

$$\sum_{i=0}^{1} a_{ki+m+1} = a_{m+1} + a_{k+m+1}$$

$$= \sum_{i=m+1-k}^{m} a_i + \sum_{i=m+1}^{k+m} a_i = \sum_{i=m+1-k}^{k+m} a_i .$$

Furthermore, if Theorem 3(a) holds for $n = p$, then it holds for $n = p + 1$ because we then have

$$\sum_{i=0}^{p+1} a_{ki+m+1} = a_{k(p+1)+m+1} + \sum_{i=0}^{p} a_{ki+m+1}$$

$$= \sum_{i=kp+m+1}^{k(p+1)+m} a_i + \sum_{i=m+1-k}^{kp+m} a_i$$

$$= \sum_{i=m+1-k}^{k(p+1)+m} a_i .$$

Hence Theorem 3(a) holds for $n \geq 1$, $m \geq 1$.

In the proof of Theorem 3(b), we apply Theorem 3(a), choosing $m = k - 1$:

$$\sum_{i=0}^{n} a_{ki+k} = \sum_{i=0}^{kn+k-1} a_i \quad .$$

Theorem 3(b) follows since the left side of this equation is equal to

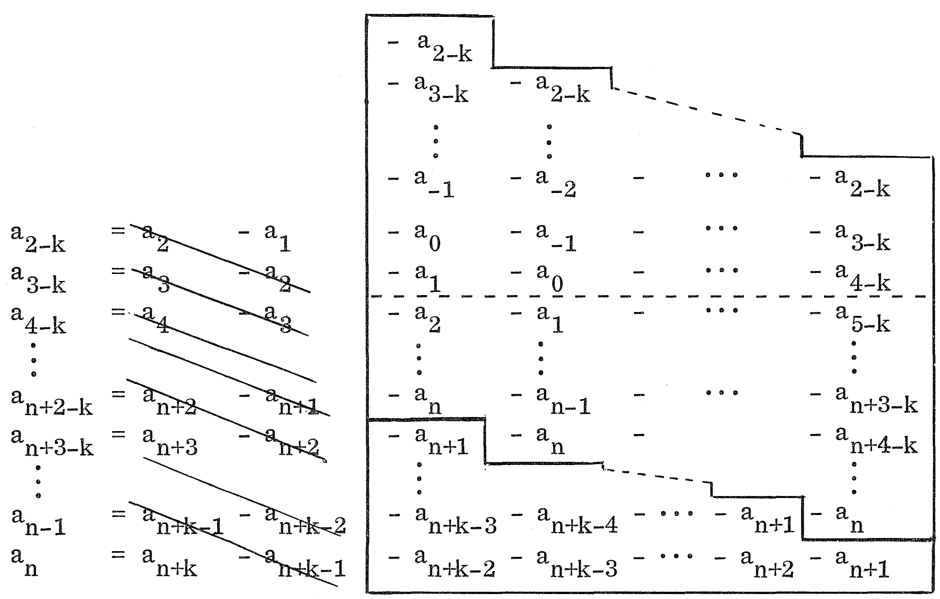
$$a_{kn+k} + \sum_{i=1}^{n} a_{ki} \quad ,$$

and the right side is equal to

$$\sum_{i=0}^{kn-1} a_i + \sum_{i=kn}^{kn+k-1} a_i = \sum_{i=0}^{kn-1} a_i + a_{kn+k} \quad .$$

Theorem (3c) is an immediate consequence of Theorem 3(b).

Inductive proofs of Theorems 4 and 5 are omitted. One may, however, verify (or discover!) Theorem 4 by considering the following diagram:



It follows from this diagram that

$$\sum_{i=2-k}^{n} a_i = a_{n+k} - a_1 - (k-2)\sum_{i=2-k}^{n} a_i + \sum_{i=1}^{k-2} ia_{-i} - \sum_{i=1}^{k-2}(k-i-1)a_{n+i} .$$

For the proof of Theorem 6, let n be any integer such that $n \geq 1$. Theorem 6 holds for $m = 1$ because

$$\sum_{j=1}^{k} a_{n-k+j}\sum_{i=1}^{j} f_{1-j+i} = \sum_{j=1}^{k}(a_{n-k+j}f_1) = a_{n+1} .$$

If Theorem 6 holds for $m = p$, then it holds for $m = p + 1$ because we then have

$$\begin{aligned} a_{n+(p+1)} = a_{(n+1)+p} &= \sum_{j=1}^{k}\left(a_{n+1-k+j}\sum_{i=1}^{j} f_{p-j+i}\right) \\ &= \sum_{j=2}^{k+1}\left(a_{n-k+j}\left\{\sum_{i=1}^{j} f_{p+1-j+i} - f_{p+1}\right\}\right) \\ &= \sum_{j=1}^{k}\left(a_{n-k+j}\sum_{i=1}^{j} f_{p+1-j+i}\right) - a_{n-k+1}f_{p+1} \\ &\quad + a_{n+1}\sum_{i=1}^{k+1} f_{p-k+i} - \left(\sum_{j=2}^{k+1} a_{n-k+j}\right) f_{p+1} \\ &= \sum_{j=1}^{k}\left(a_{n-k+j}\sum_{i=1}^{j} f_{p+1-j+i}\right) \\ &\quad + f_{p+1}(-a_{n-k+1} + 2a_{n+1} - a_{n+2}) \\ &= \sum_{j=1}^{k}\left(a_{n-k+j}\sum_{i=1}^{j} f_{p+1-j+i}\right) . \end{aligned}$$

The last equality is obtained by applying Theorem 1. Hence Theorem 6 holds for $n \geq 1$ and $m \geq 1$.

Theorem 7 obviously holds for $n = m$. We shall prove that if Theorem 7 holds for $n = mp$, then it holds for $n = m(p+1)$.

Suppose, therefore, that d_m divides a_{mp}. By Theorem 6,

$$a_{m(p+1)} = a_{mp+m} = \sum_{j=1}^{k-1}\left(a_{mp-k+j}\sum_{i=1}^{j} f_{m-j+i}\right) + a_{mp}f_{m+1} .$$

Since d_m divides each term of the sum

$$\sum_{i=1}^{j} f_{m-j+i} ,$$

where $1 \leq j \leq k-1$, and d_m divides a_{mp}, it follows that d_m divides $a_{m(p+1)}$.

For the proof of Theorem 8(a), we once again apply Theorem 6. We choose $n = 1$ and divide by f_{1+m}:

$$\frac{a_{1+m}}{f_{1+m}} = \sum_{j=1}^{k}\left(a_{1-k+j}\sum_{i=1}^{j}\frac{f_{m-j+i}}{f_{1+m}}\right) .$$

In [6] it is shown that, for any integer q,

$$\lim_{m\to\infty}\left(\frac{f_{m+q}}{f_m}\right) = r^q .$$

It follows, therefore, that

$$\lim_{n\to\infty}\left(\frac{a_n}{f_n}\right) = \sum_{j=1}^{k}\left(a_{1-k+j}\sum_{i=1}^{j} r^{i-j-1}\right)$$

$$= \frac{1}{r^k}\sum_{j=1}^{k}\left(a_{1-k+j}\sum_{i=1}^{j} r^{i-j-1+k}\right)$$

$$= \frac{1}{r^k}\sum_{j=2}^{k+1}\left(a_{j-k}\sum_{i=1}^{j-1} r^{k-i}\right) .$$

Theorem 8(b) holds since

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty}\left(\frac{a_{n+1}}{f_{n+1}}\cdot\frac{f_n}{a_n}\cdot\frac{f_{n+1}}{f_n}\right)$$

$$= \left(\lim_{n\to\infty}\frac{a_n}{f_n}\right)\left(\lim_{n\to\infty}\frac{a_n}{f_n}\right)^{-1}\left(\lim_{n\to\infty}\frac{f_{n+1}}{f_n}\right) = r\ .$$

REFERENCES

1. Brother U. Alfred, "An Introduction to Fibonacci Discovery," The Fibonacci Association, 1965.
2. Brother U. Alfred, "Exploring Recurrent Sequences," Fibonacci Quarterly, Vol. 1, No. 2, 1963, pp. 81-83.
3. M. Beberman and H. Vaughan, High School Mathematics, Course 3, D. C. Heath, 1966.
4. D. E. Ferguson, "An Expression for Generalized Fibonacci Numbers," Fibonacci Quarterly, Vol. 4, No. 3, 1966, pp. 270-272.
5. M. Feinberg, "Fibonacci-Tribonacci," Fibonacci Quarterly, Vol. 1, No. 3, 1963, pp. 71-74.
6. I. Flores, "Direct Calculation of K-Generalized Fibonacci Numbers," Fibonacci Quarterly, Vol. 5, No. 3, 1967, pp. 259-266.
7. E. P. Miles, "Generalized Fibonacci Numbers and Associated Matrices," American Mathematical Monthly, Vol. 67, No. 8, 1960, pp. 745-752.
8. N. N. Vorobyov, The Fibonacci Numbers, D. C. Heath, 1963.
9. M. E. Waddill and L. Sacks, "Another Generalized Fibonacci Sequence," Fibonacci Quarterly, Vol. 5, No. 3, 1967, pp. 209-222.

* * * * *