ARITHMETIC OF PENTAGONAL NUMBERS

RODNEY T. HANSEN
Montana State University, Bozeman, Montana

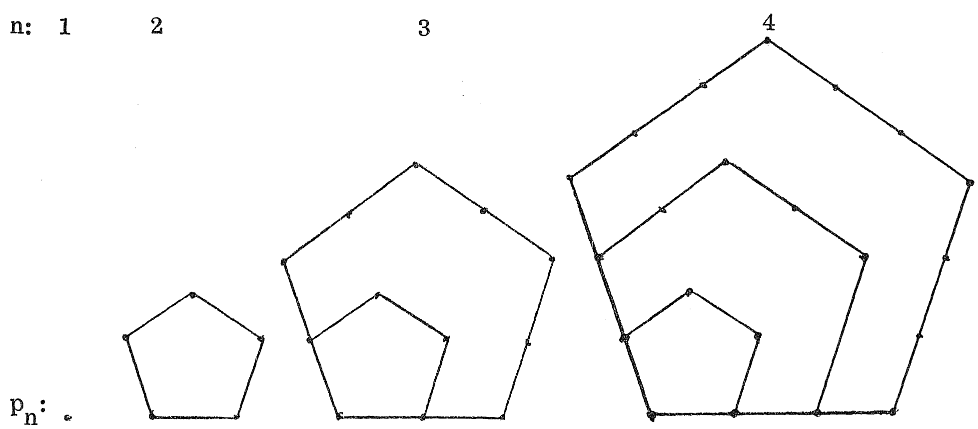
The pentagonal numbers are the integers

$$p_n = \frac{n}{2}(3n - 1), \qquad n = 1, 2, \cdots .$$

Each number p_n can also be derived by summing the first n terms of the arithmetic progression

$$1, 4, 7, 10, 13, \cdots, 3m - 2 .$$

Geometrically, considering regular pentagons homothetic with respect to one of the vertices and containing $2, 3, 4, \cdots, n$ points at equal distances along each side, the sum of all points for a given n yields p_n. Pictorially we have the following: [1, p. 10]



In this paper we shall give several algebraic identities involving pentagonal numbers of different orders. The principal result is that an infinite number of pentagonal numbers exist which are, at the same time, the sum and difference of distinct pentagonal numbers. A similar result for triangular numbers has been found by W. Sierpinski [2, pp. 31-32].

A table of p_n's will first be constructed.

p_n	0	1	2	3	4	5	6	7	8	9
0		1	5	12	22	35	51	70	92	117
1	145	176	210	247	287	330	376	425	477	532
2	590	651	715	782	852	925	1,001	1,080	1,162	1,247
3	1,335	1,426	1,520	1,617	1,717	1,820	1,926	2,035	2,147	2,262
4	2,380	2,501	2,625	2,752	2,882	3,015	3,115	3,290	3,432	3,577
5	3,725	3,876	4,030	4,187	4,347	4,510	4,676	4,845	5,017	5,192
6	5,370	5,551	5,735	5,922	6,112	6,305	6,501	6,700	6,902	7,107
7	7,315	7,526	7,740	7,957	8,177	8,400	8,626	8,855	9,087	9,322
8	9,560	9,801	10,045	10,292	10,542	10,795	11,051	11,310	11,572	11,837
9	12,105	12,376	12,650	12,927	13,207	13,490	13,776	14,065	14,357	14,652

We note from the above-given arithmetic progression that

$$p_n - p_{n-1} = 3n - 2, \qquad \text{for} \quad n = 2, 3, \cdots ,$$

and from the above table that

$$p_8 = p_4 + p_7, \quad p_{24} = p_7 + p_{23}, \quad p_{49} = p_{10} + p_{48}, \quad \text{and} \quad p_{83} = p_{13} + p_{82} .$$

Noting that the first term on the right of each of the above equalities is of the form $3n + 1$, we find that

$$p_{3n+1} = \frac{(3n + 1)}{2} [3(n + 1) - 1] = \frac{1}{2}(27n^2 + 15n + 2) .$$

Setting

$$p_m - p_{m-1} = 3m - 2 = \frac{1}{2}(27n^2 + 15n + 2)$$

we have

$$m = \frac{1}{2}(9n^2 + 5n + 2),$$

an integer. The first theorem follows.

Theorem 1. For any integer $n \geq 1$,

$$p_{\frac{1}{2}(9n^2+5n+2)} = p_{(3n+1)} + p_{\frac{n}{2}(9n+5)} .$$

A subset of the above defined pentagonal numbers yields our main result.

Theorem 2. For any positive integer n,

$$p_{\frac{1}{2}[9(3n)^2+5(3n)+2]} = p_{[3(3n)+1]} + p_{\frac{3n}{2}[9(3n)+5]}$$

$$= p_{\frac{1}{8}(6561n^4+2430n^3+495n^2+50n+8)} - p_{\frac{n}{8}(6561n^3+2430n^2+495n+50)}$$

Proof. First it is necessary to express

$$p_{\frac{1}{2}(81n^2+15n+2)}$$

in terms of n.

$$p_{\frac{1}{2}(81n^2+15n+2)} = \frac{\frac{1}{2}(81n^2 + 15n + 2)}{2}\left\{3\left[\frac{1}{2}(81n^2 + 15n + 2)\right] - 1\right\}$$

$$= \frac{1}{8}(19{,}683n^4 + 7290n^3 + 1485n^2 + 150n + 8) .$$

Equating $p_s - p_{s-1}$ to

$$p_{\frac{1}{2}(81n^2+15n+2)}$$

yields

$$s = \frac{1}{8}(6561n^4 + 2430n^3 + 495n^2 + 50n + 8) .$$

By mathematical induction on n we have that s is an integer; completing the proof.

For n = 1 and 2 we have, for example,

$$p_{49} = p_{10} + p_{48} = p_{1193} - p_{1192}$$

or

$$3577 = 145 + 3432 = 2{,}134{,}277 - 2{,}130{,}700$$

and

$$p_{178} = p_{19} + p_{177} = p_{15{,}813} - p_{15{,}812}$$

or

$$47{,}437 = 532 + 46{,}905 = 375{,}068{,}547 - 375{,}021{,}110 \; .$$

A rather curious relationship exists between n and p_{sn}; namely, that each positive integer can be expressed in an infinite number of ways as a quadratic expression involving a pentagonal number.

Theorem 3. Any positive integer n can be expressed as

$$n = \frac{1 + \sqrt{1 + 24p_{s \cdot n}}}{6 \cdot s}$$

for any positive integer s.

Proof. From the definition of a pentagonal number we have

$$p_{sn} = \frac{sn}{2}(3 \cdot sn - 1) = \frac{3(sn)^2 - sn}{2}$$

$$0 = 3s^2n^2 - sn - 2p_{sn}$$

$$n = \frac{+s \pm \sqrt{(-s)^2 - 4(3s^2)(-2p_{sn})}}{2 \cdot 3s^2} = \frac{1 \pm \sqrt{1 + 24p_{sn}}}{6s} \; .$$

Taking the positive root, the desired result is obtained.

The pentagonal numbers are not closed with respect to the operation of multiplication. However, the following three cases are quickly verified:

$$p_{87} = p_2 p_{39}, \quad p_{187} = p_4 p_{40}, \quad \text{and} \quad p_{392} = p_7 p_{47} .$$

It is not known if an infinite number of such pairs exist.

REFERENCES

1. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, New York, 1939.
2. W. Sierpiński, "Un théorème sur les nombres triangulaires," Elemente Der Mathematik, Bank 23, Nr 2 (März, 1968), pp. 31-32.

* * * * *

CORRECTIONS

Please make the following changes in "Associated Additive Decimal Digital Bracelets," appearing in the Fibonacci Quarterly in October, 1969:

On page 288, line 25, change "terms" to "forms."

On page 289, line 2, change "8" to read "B."

On page 290, line 11, change "7842" to "6842."

On page 290, line 13, change "and" to read "And."

On page 294, line 20, change "19672" to read "1967)."

On page 294, line 26, change "1969" to read "1959."

Please change the formulas given in "Diagonal Sums of Generalized Pascal Triangles," page 353, Volume 7, No. 5, December, 1969, lines 11 and 12, to read

$$p_1(q) = \sum_{k=0}^{[q/3]} \sum_{m=0}^{\left[\frac{q-3k}{2}\right]} \frac{q(q-m-2k-1)!}{(q-2m-3k)!m!k!} \cdot \left(\frac{x}{1-x}\right)^{q-m-2k}$$

$$p_2(q) = \sum_{k=0}^{[q/3]} \sum_{m=0}^{\left[\frac{q-3k}{2}\right]} \frac{q(q-m-2k-1)!}{(q-2m-3k)!m!k!} \cdot \left(\frac{x}{1-x}\right)^{q-k} (-1)^{q-m-3k}$$