SUMMATION OF POWERS OF ROOTS OF SPECIAL EQUATIONS

N. A. DRAIM
Ventura, California
and
MARJORIE BICKNELL
A. C. Wilcox High School, Santa Clara, California

It is well known that the sums of the n^{th} powers of the roots of $x^2 - x - 1 = 0$ give rise to the famous Lucas sequence $1, 3, 4, 7, 11, \cdots, L_n, \cdots,$ in which each term is the sum of the preceding two terms. In this case, one can find the roots

$$\alpha = (1 - \sqrt{5})/2,$$

and

$$\beta = (1 + \sqrt{5})/2,$$

and easily calculate $L_n = \alpha^n + \beta^n$ for $n = 1, 2, 3, \cdots$.

But what of the sums of the n^{th} powers of roots of other equations of the form

$$x^n - x^{n-1} - x^{n-2} - \cdots - x - 1 = 0\ ?$$

Soon the roots cannot be found directly but an interesting pattern of sequences of integers emerges.

The problem can be solved using symmetric functions derived in elementary theory of equations, but we prefer matrix theory. We use the following matrix properties. The trace (sum of elements on mail diagonal) of a square matrix of order n is the coefficient of x^{n-1} in its characteristic equation, when the characteristic equation is computed by subtracting x from each main diagonal element and then taking the determinant. If a matrix is raised to the k^{th} power, then its new characteristic equation has as its roots the k^{th} powers of the roots of the original equation. So, raising a matrix to successive powers and summing its main diagonal is a way of calculating sums of powers of the roots of an equation.

To sum the n^{th} powers of the roots of $x^2 - x - 1 = 0$, which is the characteristic equation of

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

simply calculate the trace for successive powers of the matrix. And for $x^3 - x^2 - x - 1 = 0$, we use

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

For

$$x^k - x^{k-1} - \cdots - x - 1 = 0,$$

write the square matrix of order k having each element in the first row equal to one, each element in the k^{th} column except the first equal to zero, and bordering an identity matrix of order $k - 1$, as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

In the table of values occurring that follows, let Σr^n signify the sum of the n^{th} powers of the roots, and

$$\begin{aligned} f(x)_1 &= x - 1 &&= 0 \\ f(x)_2 &= x^2 - x - 1 &&= 0 \\ \cdots & \quad \cdots \\ f(x)_n &= x^n - x^{n-1} - x^{n-2} - \cdots - x - 1 &&= 0 \end{aligned}$$

SUMS OF N^{th} POWERS OF ROOTS

$$f(x)_k = x^k - x^{k-1} - x^{k-2} - \cdots - x - 1 = 0$$

	1	2	3	4	5		k - 1	k
	$f(x)_1 = 0$	$f(x)_2 = 0$	$f(x)_3 = 0$	$f(x)_4 = 0$	$f(x)_5 = 0$	$\cdots$	$f(x)_{k-1} = 0$	$f(x)_k = 0$
$\sum r^0$	1	2	3	4	5	$\cdots$	$k - 1$	k
$\sum r^1$	1	1	1	1	1	$\cdots$	1	1
$\sum r^2$	1	3	3	3	3	$\cdots$	3	3
$\sum r^3$	1	4	7	7	7	$\cdots$	7	7
$\sum r^4$	1	7	11	15	15	$\cdots$	15	15
$\sum r^5$	1	11	21	26	31	$\cdots$	31	31
$\sum r^6$	1	18	39	51	57	$\cdots$	63	63
$\sum r^{k-1}$	1	L_{k-1}					$2^{k-1} - 1$	$2^{k-1} - 1$
$\sum r^k$	1	L_k						$2^k - 1$
$\sum r^{k+1}$	1	L_{k+1}						$2^{k+1} - k - 2$

$\sum r^n$ is the sum of the preceding k columnar terms.

Matric theory explains the general form given in the right column of the table. It can be proved by mathematical induction that if the nxn matrix $M = (a_{ij})$ is defined as

$$a_{ij} = \begin{cases} 1, i = 1 \text{ or } i = j + 1 \\ 0, i \neq 1 \text{ and } i \neq j + 1 \end{cases},$$

then, if $k \leq n$, $M^k = (b_{ij})$ has the following form:

$$b_{ij} = 2^{k-i} \quad \text{for } j = 1, 2, \cdots, n + i - k, \quad i = 1, 2, \cdots, n;$$
$$b_{ij} = 0 \quad \text{if } k < i \text{ and } j \geq i.$$

Thus, the trace of M^k is

$$2^{k-1} + 2^{k-2} + \cdots + 2 + 1 = 2^k - 1, \quad k \leq n.$$

Since M satisfies its own characteristic equation,

$$M^n = M^{n-1} + M^{n-2} + \cdots + M + I$$
$$M^{n+1} = M^n + M^{n-1} + \cdots + M^2 + M,$$

and the trace of M^{n+1} equals the sum of the traces of the matrices on the right, giving trace of

$$M^{n+1} = (2^n - 1) + (2^{n-1} - 1) + \cdots + (2^2 - 1) + (2^1 - 1)$$
$$= 2^{n+1} - n - 2.$$

Since finding the trace of M^k for $k \geq n$ involves summing the n preceding traces already obtained, we can form our table inductively without actually raising the matrices to powers.

For example, to get the series for $n = 5$, the sum of the five roots raised to the zero power is 5. The sum of the five roots to the first power is one, the coefficient of $-x^4$ in

$$x^5 - x^4 - x^3 - x^2 - x - 1 = 0.$$

The sum of the second, third, and fourth powers are given by

$$2^2 - 1, \; 2^3 - 1, \; 2^4 - 1.$$

The sum of the fifth powers is either $2^5 - 1$ or the sum

$$5 + 1 + 3 + 7 + 15.$$

The sum of the sixth powers is the sum of the preceding five entries,

$$1 + 3 + 7 + 15 + 31.$$