CONVOLUTION TRIANGLES FOR GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

The sequence of integers $F_1 = 1$, $F_2 = 1$, and $F_{n+2} = F_{n+1} + F_n$ are called the Fibonacci numbers. The numbers F_1 and F_2 are called the starting pair and $F_{n+2} = F_{n+1} + F_n$ is called the recurrence relation. The long division problem $1/(1 - x - x^2)$ yields

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^2 + \cdots + F_{n+1}x^n + \cdots .$$

This expression is called a generating function for the Fibonacci numbers. The generating function yielding

$$\frac{1}{(1 - x - x^2)^{k+1}} = F_1^{(k)} + F_2^{(k)}x + \cdots + F_{n+1}^{(k)} x^n + \cdots$$

is the generating function for the k^{th} convolution of the Fibonacci numbers. For $k = 0$, we get just the Fibonacci numbers. We now show two different ways to get the convolved Fibonacci numbers.

2. CONVOLUTION OF SEQUENCES

If $a_1, a_2, a_3, \cdots, a_n, \cdots$ and $b_1, b_2, b_3, \cdots, b_n, \cdots$ are two sequences, then the convolution of the two sequences is another sequence $c_1, c_2, c_3, \cdots, c_n, \cdots$ whose terms are calculated as shown:

$$\begin{aligned}
c_1 &= a_1b_1 \\
c_2 &= a_1b_2 + a_2b_1 \\
c_3 &= a_1b_3 + a_2b_2 + a_3b_1 \\
&\cdots\cdots\cdots \\
c_n &= a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots + a_kb_{n-k+1} + \cdots + a_nb_1 .
\end{aligned}$$

This last expression may also be written

$$c_n = \sum_{k=1}^{n} a_k b_{n-k+1} .$$

Let us convolve the Fibonacci number sequence with itself. These numbers we call the <u>First Fibonacci Convolution Sequence</u>:

$$\begin{aligned}
F_1^{(1)} &= F_1F_1 & &= 1 \\
F_2^{(1)} &= F_1F_2 + F_2F_1 &= 1 + 1 &= 2 \\
F_3^{(1)} &= F_1F_3 + F_2F_2 + F_3F_1 &= 2 + 1 + 2 &= 5 \\
F_4^{(1)} &= F_1F_4 + F_2F_3 + F_3F_2 + F_4F_1 &= 3 + 2 + 2 + 3 &= 10 \\
F_5^{(1)} &= \sum_{k=1}^{5} F_kF_{5-k+1} & &= 20 \\
F_6^{(1)} &= \sum_{k=1}^{6} F_kF_{6-k+1} & &= 38 \\
F_7^{(1)} &= \sum_{k=1}^{7} F_kF_{7-k+1} & &= 71 .
\end{aligned}$$

Now let us "convolve" the first Fibonacci convolution sequence with the Fibonacci sequence to get the <u>Second Fibonacci Convolution Sequence</u>:

$$\begin{aligned}
F_1^{(2)} &= F_1F_1^{(1)} &&= 1 \\
F_2^{(2)} &= F_2F_1^{(1)} + F_1F_2^{(1)} &&= 3 \\
F_3^{(2)} &= F_3F_1^{(1)} + F_2F_2^{(1)} + F_1F_2^{(1)} &&= 9 \\
F_4^{(2)} &= F_4F_1^{(1)} + F_3F_2^{(1)} + F_2F_3^{(1)} + F_1F_4^{(1)} &= 3\cdot 1 + 2\cdot 2 + 1\cdot 5 + 1\cdot 10 &= 22 \\
F_5^{(2)} &= \sum_{k=1}^{5} F_k^{(1)}F_{5-k+1} &&= 41 \\
F_6^{(2)} &= \sum_{k=1}^{6} F_k^{(1)}F_{6-k+1} &&= 111
\end{aligned}$$

The Fibonacci sequence is obtained from

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^2 + \cdots + F_{n+1}x^n + \cdots$$

The first Fibonacci convolution sequence is obtained from

$$\frac{1}{(1 - x - x^2)^2} = F_1^{(1)} + F_2^{(1)}x + F_3^{(1)}x^2 + \cdots + F_{n+1}^{(1)}x^n + \cdots .$$

The second Fibonacci convolution sequence is obtained from

$$\frac{1}{(1 - x - x^2)^3} = F_1^{(2)} + F_2^{(2)}x + F_3^{(2)}x^2 + \cdots + F_{n+1}^{(2)}x^n + \cdots .$$

These could all have been obtained by long division and continued to find as many $F_n^{(k)}$ as desired or one could have found the convoluted sequence by the method of this section. In the next section we shall see yet another way to find the convolved Fibonacci sequences.

3. THE FIBONACCI CONVOLUTION TRIANGLE

Suppose one writes down a column of zeros. To the right and one space down place a one. To get the elements below the one we add the elements one above and the one directly left. This is, of course, the rule of formation for Pascal's arithmetic triangle. Such a rule generates a convolution triangle.

Next suppose instead we add the one above and then diagonally left. Now the <u>row sums</u> are the Fibonacci numbers. We illustrate:

$$\begin{array}{rcccccccc}
 & 0 & & & & & & & \\
 & 0 & & & & & & & \\
 & 0 & 1 & & & & & & \\
 & 0 & 1 & & & & & & \\
 & 0 & 1 & 1 & & & & & \\
 & 0 & 1 & 2 & & & & & \\
 & 0 & 1 & 3 & 1 & & & & \\
 & 0 & 1 & 4 & \boxed{3} & & & & \\
 & 0 & 1 & 5 & 6 & 1 & & & \\
 & 0 & 1 & 6 & 10 & 4 & & & \\
 & 0 & 1 & 7 & \boxed{15} & 10 & 1 & & \\
\text{Column:} & 0 & 1 & 2 & 3 & 4 & \cdots & & n
\end{array}$$

However, if we add the two elements above and diagonally left, we generate the Fibonacci convolution triangle as follows. Please note these are the same numbers we got in Section 2. The zero-th column are the Fibonacci numbers, F_n; the first column are the first convolution Fibonacci numbers, $F_n^{(1)}$, etc.

	0						
	0						
	0	1					
	0	1					
	0	2					
	0	3	1				
	0	5	2				
	0	8	5				
	0	13	10	1			
	0	21	20	3			
	0	34	38	9			
	0	55	71	21	1		
	. . .	. . .	. . .	. . .	. . .		
Column:		0	1	2	3	$\cdots$	n

4. COLUMN GENERATORS OF CONVOLUTION TRIANGLES

It is easily established that the column generating functions for Pascal's triangle are

$$g_k(x) = \frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n ,$$

when the triangle is generated normally as the expansion of $(1+x)^n$, $n = 0, 1, 2, \cdots$ and as we said to do in the first part of Section 3. The column generators become

$$g_k(x) = \frac{x^{2k}}{(1-x)^{k+1}}, \qquad k = 0, 1, 2, \cdots$$

if we follow the second set of instructions. These column generators are such that the elements across the rows each are multiplied by the same power of x. We make the column move up or down by changing the power of x in the numerator of the column generating function. If we now sum

$$\sum_{k=0}^{\infty} g_k(x) = \sum_{k=0}^{\infty} \frac{x^{2k}}{(1-x)^{k+1}} = \left(\frac{1}{1-x}\right) \sum_{k=0}^{\infty} \left(\frac{x^2}{1-x}\right)^k$$

$$= \frac{1}{1-x} \cdot \frac{1}{1 - \dfrac{x^2}{1-x}} = \frac{1}{1-x-x^2} .$$

Thus the row sums across the specially positioned (Position 2) Pascal triangle are Fibonacci numbers. These are, of course, the numbers in the zero-th column of the Fibonacci convolution triangle. If we multiply the column generators of Pascal's triangle by a special set of coefficients, we may obtain other columns of the Fibonacci convolution triangle.

Recall that the k^{th} column generator of Pascal's triangle is

$$g_k(x) = \sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}} .$$

Replace x by

$$\frac{x^2}{1-x}$$

in the above to obtain

$$g_k\left(\frac{x^2}{1-x}\right) = \sum_{n=0}^{\infty} \binom{n}{k} \left(\frac{x^2}{1-x}\right)^n = \frac{\left(\dfrac{x^2}{(1-x)}\right)^k}{\left(1 - \dfrac{x^2}{1-x}\right)^{k+1}}$$

while

$$\left(\frac{1}{1-x}\right) g_k\left(\frac{x^2}{1-x}\right) = \sum_{n=0}^{\infty}\binom{n}{k} g_k = \frac{\left(\frac{1}{1-x}\right)\cdot\left(\frac{x^2}{1-x}\right)^k}{\left(1-\frac{x^2}{1-x}\right)^{k+1}} = \frac{x^{2k}}{(1-x-x^2)^{k+1}}$$

Thus the row sums are the k^{th} convolution of the Fibonacci numbers since that is the column generator we have obtained. We illustrate:

	(First column of Pascal)						
Multipliers:	0	1	2	3	4	Row Sums:	
	1						
	1						
	1	1				1	= 1
	1	2				2	= 2
	1	3	1			3·1 + 1·2	= 5
	1	4	3			4·1 + 3·2	= 10
	1	5	6	1		5·1 + 6·2 + 1·3	= 20
	1	6	10	4		6·1 + 10·2 + 4·3	= 38
	1	7	15	10	1	7·1 + 15·2 + 10·3 + 1·4	= 71
	1	8	21	20	5	8·1 + 21·2 + 20·3 + 5·4	=130

First Fibonacci Convolution Sequence

		(Second column of Pascal)						
Multipliers:	0	0	1	3	6	10	⋯	Row Sums:
	1							
	1							
	1	1						
	1	2						
	1	3	1					1
	1	4	3					3
	1	5	6	1				9
	1	6	10	4				22
	1	7	15	10	1			51
	1	8	21	20	5			111
	1	9	28	35	15	1		233

Second Fibonacci Convolution Sequence

Thus if we use the numbers in the k^{th} column of Pascal's arithmetic triangle (Position 1) as a set of multipliers with the columns of Pascal's triangle (Position 2), we get row sums which form the k^{th} Fibonacci sequence.

5. EXTENSION TO GENERALIZED FIBONACCI NUMBERS CONVOLUTION TRIANGLES

The Fibonacci numbers are the sums of the rising diagonals of Pascal's triangle which is generated by expanding $(1+x)^n$. The generalized Fibonacci numbers are defined as the sums of the diagonals of generalized Pascal's triangles which are generated by expanding

$$(1 + x + x^2 + \cdots + x^{r-1})^n .$$

The sequences can be shown to satisfy $u_1 = 1$, $u_j = 2^{j-2}$ for $j = 2, 3, \cdots, r$, and

$$u_{n+r} = \sum_{j=1}^{r} u_{n+r-j}, \quad n \geq 1 ,$$

and the generating functions are

$$\frac{1}{1 - x - x^2 - \cdots - x^{r-1}} = \sum_{n=0}^{\infty} u_{n+1} x^n .$$

The simplest instance is the Tribonacci sequence, where $T_1 = 1$, $T_2 = 1$, $T_3 = 2$, and $T_{n+3} = T_{n+2} + T_{n+1} + T_n$, and these sums are the rising diagonal sums of the expansions of $(1 + x + x^2)^n$ for $n = 0, 1, 2, 3, \cdots$. The first few terms are 1, 1, 2, 4, 7, 13, 24, 44, 81.

If we return to our Fibonacci convolution triangle at the end of Section 3, we note the row sums are the Tribonacci numbers. The column generators of the Fibonacci convolution are

$$g_k(x) = \frac{x^{3k}}{(1 - x - x^2)^{k+1}} ,$$

where the numbers on each row in the triangle all multiply the same powers of x in the column generators. Adding, we get

$$\sum_{k=0}^{\infty} g_k(x) = \left(\frac{1}{1 - x - x^2}\right) \sum_{k=0}^{\infty} \left(\frac{x^3}{1 - x - x^2}\right)^k = \frac{1}{1 - x - x^2 - x^3}$$

which is the Tribonacci sequence generating function. If we use the special multipliers $\binom{n}{k}$ as before, we get

$$\left(\frac{1}{1 - x - x^2}\right) \sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{x^3}{1 - x - x^2}\right)^k = \sum_{k=0}^{\infty} \binom{n}{k} g_k(x) = \frac{x^{3k}}{(1 - x - x^2 - x^3)^{k+1}}$$

and this is the k^{th} Tribonacci convolution sequence generator and the coefficients appear in the k^{th} column of the Tribonacci convolution triangle. Thus we can obtain all the columns of the Tribonacci convolution triangle from the Fibonacci convolution triangle in the same way we obtained the Fibonacci convolution triangle from Pascal's arithmetic triangle.

We can thus generate a sequence of convolution triangles whose zero-th columns are the rising diagonal sums taken from generalized Pascal triangles induced from expansions of $(1 + x + x^2 + \cdots + x^{r-1})^n$. The column generators for the r^{th} case

$$g_k(x) = \frac{x^{rk}}{(1 - x - x^2 - \cdots - x^{r-1})^{k+1}}$$

can easily be seen to generate the column generators for the $(r+1)^{st}$ case

$$g_k(x) = \frac{x^{(r+1)k}}{(1 - x - x^2 - \cdots - x^r)^{k+1}}$$

using the preceding methods.

Referring back to the Fibonacci convolution triangle of section three, each number in the triangle is the sum of the one number above and the number diagonally left. Because the column generators must obey that law and multiplying by powers of x so that the proper coefficients will be added, we could write a recurrence relation for the column generators of the Pascal convolution triangle as follows:

$$G_k(x) = xG_k(x) + x^2G_{k-1}(x) \quad \text{or} \quad G_k(x) = \frac{x^2}{1-x}G_{k-1}(x) .$$

By similar reasoning, each number of the Fibonacci convolution triangle is the sum of the two terms above it and one diagonally left. Proceeding to column generators, then,

$$G_k(x) = xG_k(x) + x^2G_k(x) + x^3G_{k-1}(x)$$

or

$$G_k(x) = \frac{x^3}{1 - x - x^2} G_{k-1}(x) .$$

6. THE REVERSE PROCESS

One can retrieve the Fibonacci convolution triangle from the Tribonacci convolution triangle quite simply. First recall

$$\sum_{n=0}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1 - x)^{k+1}} .$$

Replace x by $-x$; then it becomes

$$\sum_{n=0}^{\infty} \binom{n}{k} (-1)^n x^n = \frac{(-1)^k x^k}{(1 + x)^{k+1}} ,$$

or

$$\sum_{n=0} \binom{n}{k} (-1)^{n+k} x^n = \frac{x^k}{(1+x)^{k+1}} .$$

With these multipliers,

$$\binom{n}{k} (-1)^{n+k} ,$$

we can return from Tribonacci to Fibonacci.

Let the column generators of the Tribonacci case be

$$g_n(x) = \frac{x^{3n}}{(1 - x - x^2 - x^3)^{n+1}} ,$$

and multiplying through by

$$\binom{n}{k} (-1)^{n+k} ,$$

and summing, yields

$$\sum_{n=0}^{\infty} \binom{n}{k} (-1)^{n+k} g_n(x) = \frac{1}{1 - x - x^2 - x^3} \sum_{n=0}^{\infty} \binom{n}{k} (-1)^{n+1} \left(\frac{x^3}{1 - x - x^2 - x^3}\right)^n$$

$$= \frac{x^{3k}/(1 - x - x^2 - x^3)^{k+1}}{\left(1 + \frac{x^3}{1 - x - x^2 - x^3}\right)^{k+1}} = \frac{x^{3k}}{(1 - x - x^2)^{k+1}}$$

which are the column generators of the Fibonacci convolution triangle. The same thing applies, in general, to return from the $(r+1)^{st}$ convolution triangle to the r^{th} convolution triangle.

7. SPECIAL PROBLEMS

1. Assuming Pascal's triangle in Position 1 and the column generators are

$$g_k(x) = \frac{x^k}{(1 - x)^{k+1}} ,$$

then show the row sums of Pascal's triangle are the powers of 2. Hint:

$$\frac{1}{1 - 2x} = \sum_{n=0}^{\infty} 2^n x^n .$$

2. Assuming the Fibonacci convolution triangle has its columns positioned so that

$$g_k(x) = \frac{x^k}{(1 - x - x^2)^{k+1}} ,$$

then show the row sums are the Pell numbers $P_1 = 1$, $P_2 = 2$, $P_{n+2} = 2P_{n+1} + P_n$. Hint:

$$\frac{1}{1 - 2x - x^2} = \sum_{n=0}^{\infty} P_{n+1}x^n .$$

3. Show that the convolution triangle for the sequence $1, 3, 3^2, \cdots, 3^n, \cdots$ can be obtained from the convolution triangle for the sequence $1, 2, 2^2, 2^3, \cdots, 2^n, \cdots$ using the techniques discussed in this paper.

4. By using the coefficients in

$$\sum_{n=0}^{\infty} \binom{n}{k} (-1)^{n+k} x^n ,$$

as multipliers, show how to get the convolution triangle for the alternate Fibonacci numbers from the convolution triangle for the powers of three. Hint:

$$\frac{1}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1}\, x^n \;.$$

5. By using the multipliers from

$$\sum_{n=0}^{\infty} 2\left[\binom{n}{k} \cdot 3^n\right] x^n \;,$$

on the Fibonacci convolution triangle with column generators

$$g_k(x) = \frac{x^k}{1 - x - x^2} \;,$$

obtain the convolution triangle for every third Fibonacci number sequence. Hint:

$$\frac{2}{1 - 4x + x^2} = \sum_{n=0}^{\infty} F_{3n+1}\, x^n \;.$$

8. OTHER CONVOLUTION TRIANGLES

Let

$$\left(\frac{(1 - x)^{q-1}}{(1 - x)^q - x^{p+q}}\right)^{k+1} = \sum_{n=0}^{\infty} u^{(k)}(n;\, p, q)\, x^n \;,$$

be the k^{th} convolution of the sequence $u(n;\, p,q)$, where the sequences $u(n;\, p,q)$ are the generalized Fibonacci numbers of Harris and Styles [1]. (Also see [2].)

Let

$$g_n(x) = \frac{x^{(p+q)n}}{(1 - x)^{nq+1}}$$

$$\sum_{n=0}^{\infty} g_n(x) = \frac{1}{1 - x} \sum_{n=0}^{\infty} \left(\frac{x^{p+q}}{(1 - x)^q} \right)^n = \frac{(1 - x)^{q-1}}{(1 - x)^q - x^{p+q}} \; .$$

But,

$$\sum_{n=0}^{\infty} \binom{k + n}{k} x^n = \frac{1}{(1 - x)^{k+1}} \; .$$

Thus,

$$\frac{1}{1 - x} \sum_{n=0}^{\infty} \binom{k + n}{k} \left(\frac{x^{p+q}}{(1 - x)^q} \right)^n = \frac{(1 - x)^{(k+1)q-1}}{\left[(1 - x)^q - x^{p+q}\right]^{k+1}}$$

$$\sum_{n=0}^{\infty} \binom{k + n}{k} g_n^{(k)}(x) = \frac{1}{(1 - x)^{k+1}} \sum_{n=0}^{\infty} \binom{k + n}{n} \left(\frac{x^{p+q}}{(1 - x)^q} \right)^n$$

$$= \frac{(1 - x)^{(k+1)q-1-k}}{\left[(1 - x)^q - x^{p+q}\right]^{k+1}}$$

$$= \frac{(1 - x)^{(q-1)(k+1)}}{\left[(1 - x)^q - x^{p+q}\right]^{k+1}} = \sum_{n=0}^{\infty} u^{(k)}(n; p, q) x^n$$

and the $g_n^{(k)}(x)$ are the corresponding column generators in the Pascal's triangle with the first k columns trimmed off.

9. REVERSING THE PROCESS, AGAIN

If we consider the convolution triangles whose column generators are

$$g_n(x) = \frac{(x^{p+q})^n}{\left((1-x)^q - x^{p+q}\right)^{n+1}},$$

and if we sum these with alternating signs,

$$\sum_{n=0}^{\infty} (-1)^n g_n(x) = \frac{1}{(1-x)^q - x^{p+q}} \; \frac{1}{1 + \dfrac{x^{p+q}}{(1-x)^q - x^{p+q}}} = \frac{1}{(1-x)^q}$$

while

$$\left[\frac{1}{(1-x)^q - x^{p+q}}\right]^k \sum_{k=0}^{\infty} \binom{n+k}{k} (-1)^n g_n(x) = \frac{1}{(1-x)^{q(k+1)}} .$$

Thus, we can recover the columns of Pascal's triangle from the above convolution triangle. This may be extended in many ways. Thus, we can obtain the convolution triangles for all the sequences $u(n; p,q)$ by using multipliers from Pascal's triangle on the column generators of Pascal's triangle and taking row sums.

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