LINEAR RECURSION RELATIONS — LESSON EIGHT
ASYMPTOTIC RATIOS IN RECURSION RELATIONS

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One of the marvels associated with Fibonacci sequences is the fact that for all such sequences the limit of the ratio T_{n+1}/T_n as n approaches infinity is the Golden Section Ratio

$$\frac{1+\sqrt{5}}{2} .$$

The following table shows the ratio of successive terms for the Fibonacci sequence 2, 5, 7, 12, 19, 31, ··· .

n	T_n	T_n/T_{n-1}
1	2	
2	5	
3	7	1.4000000
4	12	1.7142857
5	19	1.5833333
6	31	1.6315789
7	50	1.6129032
8	81	1.6200000
9	131	1.6172839
10	212	1.6183206
11	343	1.6179245
12	555	1.6180758
13	898	1.6180180
14	1453	1.6180400
15	2351	1.6180316
16	3804	1.6180348
17	6155	1.6180336
18	9959	1.6180341
19	16114	1.6180339

But is this indeed so remarkable? There are many other sequences which have limiting ratios and likewise some in which there is no limit. For example, in the Tribonacci Sequence: 1, 2, 4, 7, 13, ··· where the last three terms are added together to get the next term, successive ratios are as shown terms are added together to get the next term successive ratios are as shown in the following table.

n	T_n	T_n/T_{n-1}
1	1	
2	2	
3	4	
4	7	1.7500000
5	13	1.8571428
6	24	1.8461538
7	44	1.8333333
8	81	1.8409090
9	149	1.8395061
10	274	1.8389261
11	504	1.8394160
12	927	1.8392857
13	1705	1.8392664

A recursion relation: $T_{n+1} = 3T_n - 4T_{n-1}$ yields a sequence which does not have a limiting ratio. For example, if $F_1 = 5$, $T_2 = 9$, the ratios are as shown in the following table.

n	T_n	T_n/T_{n-1}
1	5	
2	9	
3	7	0.7777777
4	-15	-2.1428571
5	-73	4.8666666
6	-159	2.1780821
7	-185	1.1635220
8	81	-0.4378378
9	983	12.1358024
10	2625	2.6703967
11	3943	1.5020952
12	1329	0.3370530
13	-11785	-8.8675696

Clearly, several questions emerge:

1. Which sequences have a limiting ratio?
2. Which sequences do not have a limiting ratio?
3. On what does the limiting ratio depend?

These questions can be answered conveniently on the basis of expressing T_n in terms of the roots of the auxiliary equation

THE FIBONACCI SEQUENCE

Consider the sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, $\cdots$. Here,

$$F_n = \frac{r^n - s^n}{\sqrt{5}} ,$$

where

$$r = \frac{1 + \sqrt{5}}{2} = 1.61803 \cdots ,$$

and

$$s = \frac{1 - \sqrt{5}}{2} = -0.61803 \cdots .$$

The

$$\lim_{n \to \infty} F_n / F_{n-1} = \frac{r^n - s^n}{r^{n-1} - s^{n-1}} .$$

Dividing the terms of numerator and denominator by the $(n-1)^{st}$ power of r, this ratio takes the form

$$\lim_{n \to \infty} \frac{r - s(s/r)^{n-1}}{1 - (s/r)^{n-1}} .$$

Since the absolute value of s/r is less than 1, the limit of the $(n-1)^{st}$ power of this ratio as n goes to infinity is zero. Thus

$$\lim_{n \to \infty} F_n / F_{n-1} = r .$$

A similar analysis can be made for any Fibonacci sequence. We have found that for such a sequence,

$$T_n = Ar^n + Bs^n .$$

Therefore,

$$\lim_{n\to\infty} T_{n+1}/T_n = \frac{Ar^{n+1} + Bs^{n+1}}{Ar^n + Bs^n}$$

$$= \lim_{n\to\infty} \frac{r + (B/A)s(s/r)^n}{1 + (B/A)(s/r)^n} = r \ .$$

One thing we can learn from this analysis is that the root with larger absolute value, r, dominates the root with smaller absolute value, s.

REAL AND UNEQUAL ROOTS

Clearly, if

$$T_n = Ar^n + Bs^n + Ct^n \cdots \ ,$$

where the roots are real and unequal and $r \quad s \quad t \cdots$ then the limiting ratio of T_{n+1}/T_n will be r.

EQUAL AND REAL ROOTS

If some of the roots are equal, but there is another real root which has the largest absolute value, this latter root will dominate to give the limiting ratio in the sequence. If the equal roots have the largest absolute value, then (consider three equal roots, r).

$$T_n = (An^2 + Bn + C)r^n + Ds^n + Et^n \cdots \ .$$

Therefore $\lim_{n\to\infty} T_{n+1}/R_n$ will equal

$$\lim_{n\to\infty} \frac{\left\{A(n+1)^2 + B(n+1) + C\right\} r^{n+1} + Ds^{n+1} + Et^{n+1} \cdots}{An^2 + Bn + C \ r^n + Ds^n + Et^n \cdots}$$

$$\lim_{n\to\infty} \frac{\left\{(n+1)^2/n^2 + (B/A)(n+1)/n^2 + C/(An^2)\right\} r + (Ds/An^2)(s/r)^n \cdots}{1 + B/(An) + C/(An^2) + (D/An^2)(s/r)^n \cdots} = r$$

Thus the dominant real root again determines the limit of the sequence ratio.

COMPLEX ROOTS

For the type of linear recursion relation we are considering in which the coefficients are real numbers, the complex roots of the auxiliary equation occur in conjugate pairs. Let the portion of T_n dependent on these roots be given by

$$cr^n + c'(r')^n ,$$

where c and c' are complex conjugate coefficients. Now set:

$$c = Ce^{\lambda i} \quad \text{and} \quad c' = Ce^{-\lambda i}$$
$$r = Re^{\phi i} \quad \text{and} \quad r' = Re^{-\phi i}$$

where C and R are the absolute values of the complex quantities c and r, respectively. Then

$$cr^n + c'(r')^n = CR^n e^{(\lambda + n\phi)i} + CR^n e^{-(\lambda + n\phi)i}$$
$$= 2CR^n \cos(\lambda + n\phi) .$$

If there is a real root with greater absolute value than R, this real root will dominate and the sequence ratio will converge. However, if R is greater than any of the real roots, it will dominate them. Only the cosine factor involving n will not converge either directly or in ratio. Thus a sequence in which there is a pair of complex roots whose absolute value is greater than the absolute value of any of the real roots will be a sequence without a limiting ratio.

A COROLLARY

Suppose we are seeking the roots of the cubic

$$x^3 - 7x^2 + 8x - 4 = 0 .$$

From one point of view this might be looked upon as the auxiliary equation of the recursion relation

$$T_{n+1} = 7T_n - 8T_{n-1} + 4T_{n-2} .$$

If we then calculate the terms of a sequence obeying this relation and find that their ratio approaches a limit with increasing n, this limiting ratio would correspond to the largest real root of the cubic. In the present instance, this ratio comes out to be 5.7245767.

PROBLEMS

1. Using the ratio of successive terms of a sequence, determine the largest real root of the equation: $x^3 - 12x^2 + 9x - 7 = 0$.

2. By analyzing the roots of the auxiliary equation, determine the limiting ratio of successive terms in the sequences obeying the recursion relation: $T_{n+1} = 8T_{n-1} + 3T_{n-2}$.

3. By analyzing the roots of the auxiliary equation, determine the limiting ratio of successive terms of sequences having the recursion relation: $T_{n+1} = -3T_n + T_{n-1} + 8T_{n-2} + 4T_{n-3}$.

4. If $R_n = 5(-1)^n$ and $S_n = F_n$, what is the limiting ratio of terms of the sequence $T_n = R_n + S_n$?

5. If

$$R_n = 2^n(n^2 + 3n + 5) \text{ and } S_n = 3S_{n-1} + S_{n-2} ,$$

with $S_1 = 1$, $S_2 = 5$, find the limiting ratio of $T_n = R_n + S_n$.

6. By analyzing the auxiliary equation, show that the recursion relation

$$T_{n+1} = 3T_n - 7T_{n-1} + 10T_{n-2}$$

governs sequences which do not have a limiting ratio.

Solutions to problems may be found on page 324.