MOSAIC UNITS: PATTERNS IN ANCIENT MOSIACS

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Inspecting ancient floor mosaics, I noticed [1] that their geometric patterns tend to fall into the same few size groups, despite the mosaics being in widely separated parts of the classical world.

On measuring all alternative dimensions that it seemed reasonable to measure on each pattern, and doing this for many patterns of the same size group, I obtained a histogram as in Fig. 1.

In every size group I obtained the same basic pattern of histogram; some little peaks followed by a very tall peak, followed by a succession of diminishing waves of small peaks. Examination of these histograms revealed that nearly every pattern has one dimension contributing to the very tall peak. This dimension can be said to be common to every pattern in the size group concerned.

That virtually every pattern of a size group has one dimension of virtually (i. e., within the spread of the very tall peak) the same length, suggests that this dimension was fixed by the mosaicists. Lack of many alternative dimensions would explain why patterns fall into size groups.

Examining equal pattern dimensions on different mosaics, I found that they are not composed of equal numbers of stones. Even on the same mosaic, constant dimensions are often composed of varied numbers of stones. Mosaicists fixing dimensions by measurement, rather than by counting out stones, would explain this.

Measuring (with a class interval of 1 millimeter) 121265 dimensions of patterns that had apparently been originally fixed by mosaicists' measurements, I obtained a frequency distribution as in Figure 2. Measuring more patterns, to a total of more than 310000 pattern dimensions, I found essentially the same distribution; the very tall peaks grew much taller, whilst some extra, but minute, peaks appeared. From the distribution (Fig. 2), it is clear that ancient floor mosaic geometric patterns are remarkably few different absolute sizes. Can we account for it?

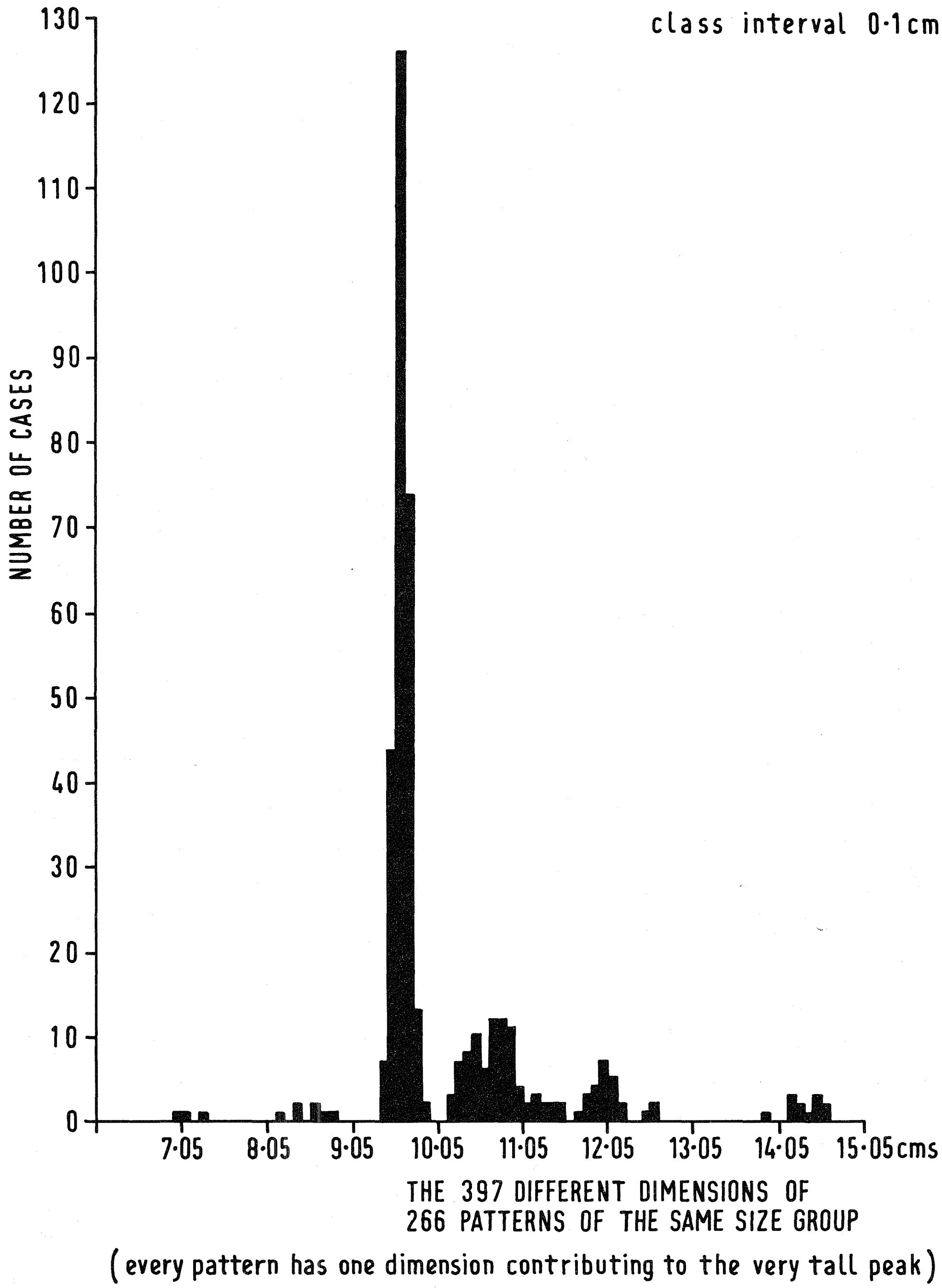
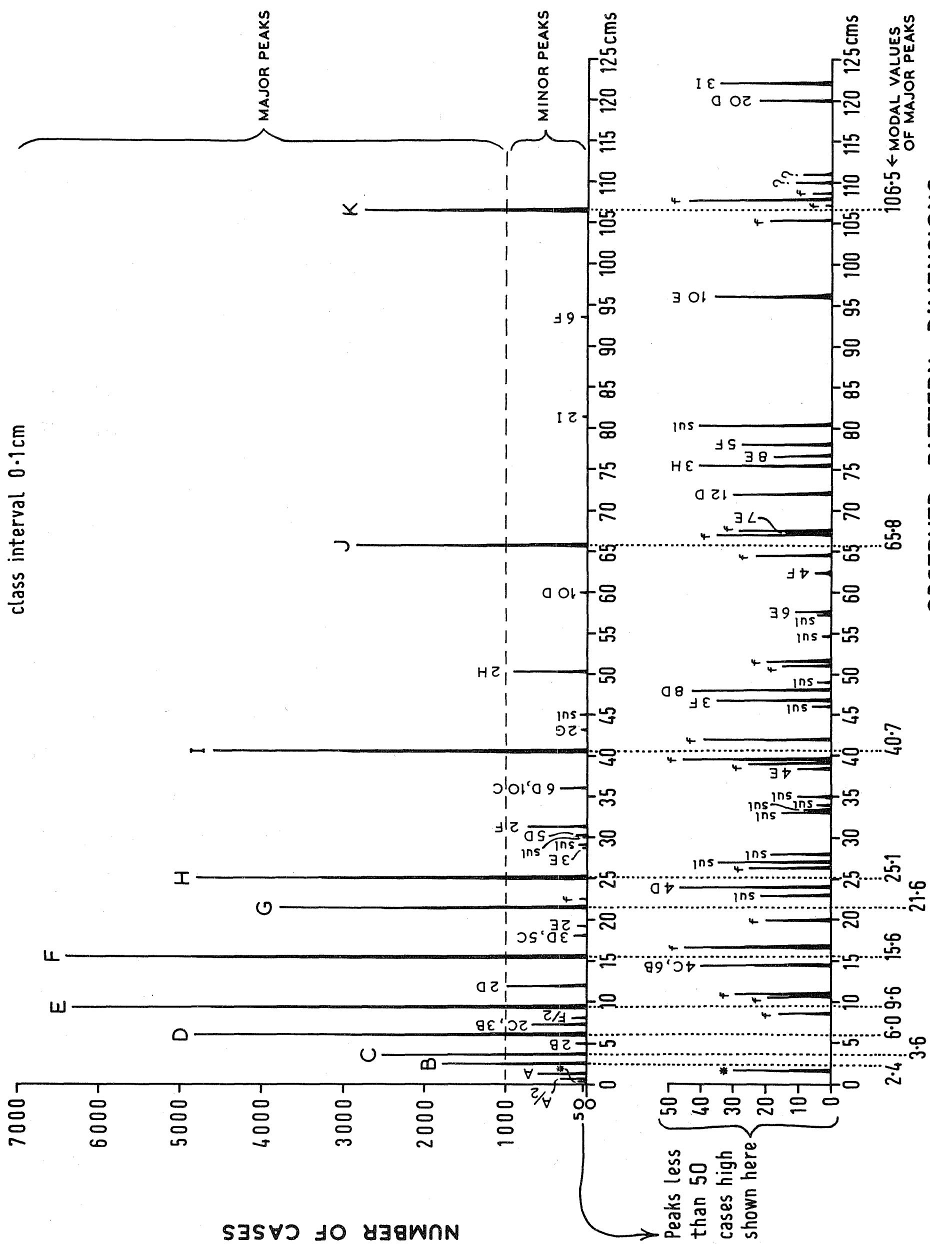

class interval 0·1cm
NUMBER OF CASES
130
120
110
100
90
80
70
60
50
40
30
20
10
0
7·05
8·05
9·05
10·05
11·05
12·05
13·05
14·05
15·05cms
THE 397 DIFFERENT DIMENSIONS OF
266 PATTERNS OF THE SAME SIZE GROUP
(every pattern has one dimension contributing to the very tall peak)

Figure 1



class interval 0·1cm
MAJOR PEAKS
MINOR PEAKS
NUMBER OF CASES
Peaks less than 50 cases high shown here
OBSERVED PATTERN DIMENSIONS
MODAL VALUES OF MAJOR PEAKS
2·4
3·6
6·0
9·6
15·6
21·6
25·1
40·7
65·8
106·5
125cms

Figure 2

In the frequency distribution (Fig. 2), there are only 10 peaks higher than 1000 cases. For purposes of analysis, I call these major peaks (labelled B, C, D, ···, Fig. 2). Many of the remaining minor peaks are so small that they have to be shown on an enlarged scale — lower part of Fig. 2.

Peaks occurring at twice or three times the length represented by another peak may owe their existence to just this, i.e., mosaicists having used two or three times a measuring unit, the latter represented by the "basic" peak. Many of the observed peaks exhibit this property. That they arose as being multiples of a more basic unit would be reinforced if peaks which are multiples of another are small peaks, whilst the peak of which they are a multiple is a much higher peak. Many of the minor peaks lie at lengths that are whole multiples of the lengths represented by the major peaks. I regard these as probably having arisen in this way (marked accordingly, Fig. 2).

The values of many ancient standard units of length have come down to us, so it is possible to see whether any observed peaks coincide with known ancient units. Some do, but surprisingly, only a few minor peaks agree with known standard units (marked "s u l," Fig. 2).

We are now left with the major peaks and a few minor ones. Some of the latter are caused by me measuring pattern dimensions which happen to also be the widths of single mosaic stones (marked with an asterisk, Fig. 2). The remaining minor peaks (with the exception of the one at 1.2 cm) have the common property of lying adjacent to one or other of the major peaks (two lie adjacent to one of the tallest minor peaks). Identification of the pattern dimensions that these minor peaks (marked "f," Fig. 2) represent, shows that they account for virtually every case of the few instances where I was unable to decide which of two alternative measurements was the one I should measure, in the sense of trying to measure the distance most likely to have been set down by measurement by the mosaicists. One of these two alternatives must be wrong, in the sense that they cannot both be right. In all but two cases, the alternative measurement lies in the adjacent major peak. That it should coincide with the dimension that the majority of pattern sizes exhibit, is reason to consider this value as the true one. On the other hand, we could reject both alternative measurements. It will not affect the results, for they account for less than 0.85 percent of the observations.

I find unusually wide stones are often associated [2] with distortion of the arrangement of neighboring stones. Construction can be deduced to have proceeded from the unusual stone through the area it distorts. Making maps of such effects leads me to think [2] that mosaics were normally started at their center, and constructed progressively outwards from it.

Assigning imperfections in mosaics values on a numerical scale of increasing imperfection [2] usually yields a map as in Figure 3. Assuming imperfections increase as construction progresses, this again indicates that construction was centrifugal, but also that it was fastest in the four axial directions (A, B, C, D, Fig. 3).

Consequently, the first parts of patterns to be reached in construction would be their parts nearest to the mosaic center (their innermost rim, for patterns centered on the mosaic center) and the first of these parts to be reached will be the part lying on the mosaic axis. Constructional measurements would thus presumably have been made primarily in the mosaic axes and to the inner rims of patterns.

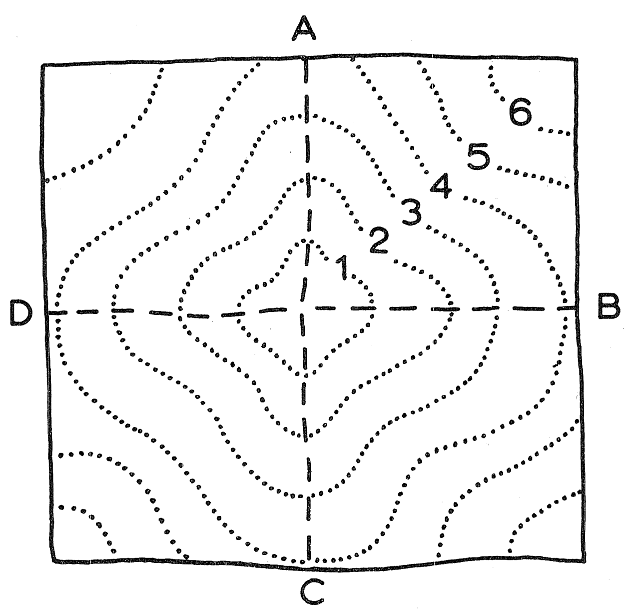


Figure 3

That the ancients usually proceeded like this is supported by the typical shape of the histogram as in Fig. 1. The dimension I deduce as being the one that the mosaicists made (because it lies in the very tall peak (Fig. 1)), is also usually the dimension that I had measured to the inner rim of the pattern. The other alternative dimensions of each pattern, in the vast majority of cases, all lie to the right of this tall peak (Fig. 1). These, representing greater lengths, are those I measured mostly to outer rims of patterns. The rarity of cases where the mosaicists' measurement was apparently not to the inner rim of the pattern, is shown by the scarcity of observations to the left of the very tall peak (Fig. 1).

The crests of the waves of peaks following the very tall peak (Fig. 1) lie at intervals which agree with the lengths represented by the tall peak in each of the smaller size groups of patterns. Consequently, since these crests are caused by including the pattern "thickness" in the measurement, this reveals that pattern thicknesses were often also fixed in terms of the same units as were used to fix the sizes of the smaller patterns.

Resuming analysis of Fig. 2, we are left with the major peaks and a minor peak at 1.2 cm. The modal values of the major peaks are: 2.4, 3.6, 6.0, 9.6, 15.6, 21.6, 25.1, 40.7, 65.8, and 106.5 cm, respectively. Of the 310000 pattern dimensions, 89% lie within ±3 standard deviations ($\sigma \equiv 0.13$ cm for each major peak) of these values. (A further 9% lie at whole multiples of these values. Of the remaining 2%, only approximately 1% can be identified with known standard units of length.)

Presumably we can regard these ten values, responsible for 89% of the observed lengths, as the units that were marked on the rulers which Vitruvius (first century B.C.) tells us [3] that mosaicists "accurately used." I call these values <u>mosaic units</u>.

DETERMINATION OF MORE ACCURATE VALUES FOR MOSAIC UNITS

That it is right to regard mosaic units as a set, is suggested by them lying in a distinct series (ignoring 21.6 cm); each is virtually the sum of the preceding two. On this basis, we might expect, by extrapolation, larger pattern sizes of 172.8, 279.6, and 452.4 cm. I find that the typical pattern sizes greater than 106.5 cm do occur very nearly at these distances, but fall progressively slightly short of these expected values (Fig. 4).

Observed modal value	Hypothetical value k = 1·2 cm	Difference between observed and hypothetical values	Number of cases	Relative reliability of mode
1·2 cm	1·2 cm	0 cm	1313	x 4
2·4	2·4	0	5031	x 7
3·6	3·6	0	6298	x 8
6·0	6·0	0	13970	x 12
9·6	9·6	0	15231	x 12
15·6	15·6	0	16150	x 13
21·6	–	–	10668	x 10
25·1	25·2	0·1	14617	x 12
40·7	40·8	0·1	12785	x 11
65·8	66·0	0·2	5861	x 8
106·5	106·8	0·3	3256	x 6
172·3	172·8	0·5	1553	x 4
278·9	279·6	0·7	426	x 2
451·3	452·4	1·1	485	x 3
730·2	732·0	1·8	144	unity
~1180·3	1184·4	~4·1	>10	–
unknown	–	–	–	–

Figure 4

Unfortunately, very large patterns are rare, for there are few mosaics big enough to exhibit them. I do not yet have sufficient observations to confidently report a value for the observed pattern size corresponding to the expectation of 1184.4 cm, but an approximate observed value is 1180.3 cm.

The two smallest lengths represented by major peaks are 3.6 and 2.4 cm. Extrapolating the series in this direction, we obtain 2.6 - 2.4 = 1.2 cm. This prediction is confirmed by the minor peak at 1.2 cm. Attempting to extrapolate again, we get 2.4 - 1.2 = 1.2 cm, demonstrating that 1.2 cm can be regarded as the basis of the set of mosaic units.

If mosaic units were in fact each the sum of the preceding two, that hypothetical values based on a value of 1.2 cm for the first unit progressively exceed the longer observed lengths by slightly greater amounts (Fig. 4) suggests that the true starting value is slightly less than 1.2 cm. The value of the first mosaic unit (M_1) in the series $M_x = M_{x-1} + M_{x-2}$ which yields values with the best fit to the observations can be determined as follows.

If each unit is the sum of the preceding two, the series can be expressed by the Fibonacci numbers, taking the first value as unity. To give values in a particular system of measure, I introduce a constant k equal to the dimension of the first value in the units of measure desired. A generating relation for mosaic units is therefore:

$$(1) \qquad y = k\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^x - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^x\right]$$

The expression in square brackets yielding Fibonacci numbers by successive substitution of integers 1, 2, 3, $\cdots$ for x; and y, the value of the x^{th} mosaic unit, assumes the units of measure of k.

The observed modal mid-interval value of the first mosaic unit is 1.2 cm. Its true value probably lies somewhere in the range of this modal value: 1.15 $\rightarrow$ 1.25 cm caused by the class interval of 0.1 cm.

From Eq. (1), division of each observed mosaic unit by its pertinent Fibonacci number gives a value for k. The larger the value that this is done for, the more accurate the result. For the first six mosaic units k = 1.2. For units 25.1, 40.7, and 65.8 cm, k begins to be slightly less than 1.2, and for the five units bigger than 65.8 cm, $k = 1.197$ cm for each. It thus appears that k is less than 1.2 cm, probably about 1.197 cm.

Hypothetical values based on 1.197 are shown in Figure 5. Also shown are these values corrected to the nearest whole millimeter, so as to bring them to a form comparable with the observations (class interval 1 mm). In all but one case (Fig. 5), values based on 1.197 cm match the observations.

Trying $k = 1.196$ cm and $k = 1.198$ cm, in both cases the resulting theoretical values for mosaic units progressively diverge from the observed values. Moreover (Figure 6), they diverge in an approximately symmetrical way, indicating that 1.197 cm represents the best value (in cms, to three places of decimals) for k.

A value for k yielding values agreeing with all the observed mosaic units is impossible. Taking $k = 1.197$ cm gives values fitting all observations (ignoring 21.6 cm) except 172.3 cm, for which the theoretical value is 172.36 cm ($\equiv$ 172.4 cm). The smallest change in 172.368 cm needed to make it fall into the same class interval as the observed value (172.3 cm) is 0.024 cm. The Fibonacci number for this unit is 144. Thus the necessary change in k is (0.024/144) cm = 0.0001666 cm. This gives a new set of hypothetical values for mosaic units, but whilst fitting the observation 172.3 cm, it begins to diverge from the observations at the 14th and 15th mosaic unit (Fig. 5).

If the first mosaic unit was 1.197 cm long, it explains why some observed mid-interval values appear to be only approximately the sum of the preceding two. For example, we have the observed values 9.6, 15.6, 25.1 cm, but 9.6 + 15.6 = 25.2, not 25.1. However, based on 1.197 cm, we have 9.576 + 15.561 = 25.137 which is exact. Rounding each to the nearest whole millimeter (which is the effect of the class interval of 1 mm), we get

$$9.576 \ (\equiv 9.6) + 15.561 \ (\equiv 15.6) = 25.137 \ (\equiv 25.1)$$

which explains this.

We might expect a similar effect in pattern sizes that are multiples of others. In some cases, this is so; for example, a minor peak occurs (Fig. 2) with modal value 50.3 cm. This could be caused by use of 2x unit 25.1 = 50.2 cm. If the true value of the eighth mosaic unit is 25.137 cm, we get

Observed modal value	Hypothetical value k = 1·197 cm	Previous column correct to nearest whole mm	Difference between previous column and observed value	Hypothetical value k = 1·1968334 cm	Previous column to nearest whole mm, minus observed value
1·2 cm	1·197 cm	1·2 cm	0 cm	1·1968334 cm	0 cm
2·4	2·394	2·4	0	2·3936668	0
3·6	3·591	3·6	0	3·5905002	0
6·0	5·985	6·0	0	5·9841670	0
9·6	9·576	9·6	0	9·5746672	0
15·6	15·561	15·6	0	15·5588342	0
21·6	—	—	—	—	—
25·1	25·137	25·1	0	25·1335014	0
40·7	40·698	40·7	0	40·6923356	0
65·8	65·835	65·8	0	65·8258370	0
106·5	106·533	106·5	0	106·5181726	0
172·3	172·368	172·4	0·1	172·3440096	0
278·9	278·901	278·9	0	278·8621822	0
451·3	451·269	451·3	0	451·2061918	−0·1
730·2	730·170	730·2	0	730·0683740	+0·1
~1180·3	1181·439	1181·4	~1·1	—	—
unknown	1911·609	—	—	—	—

Figure 5

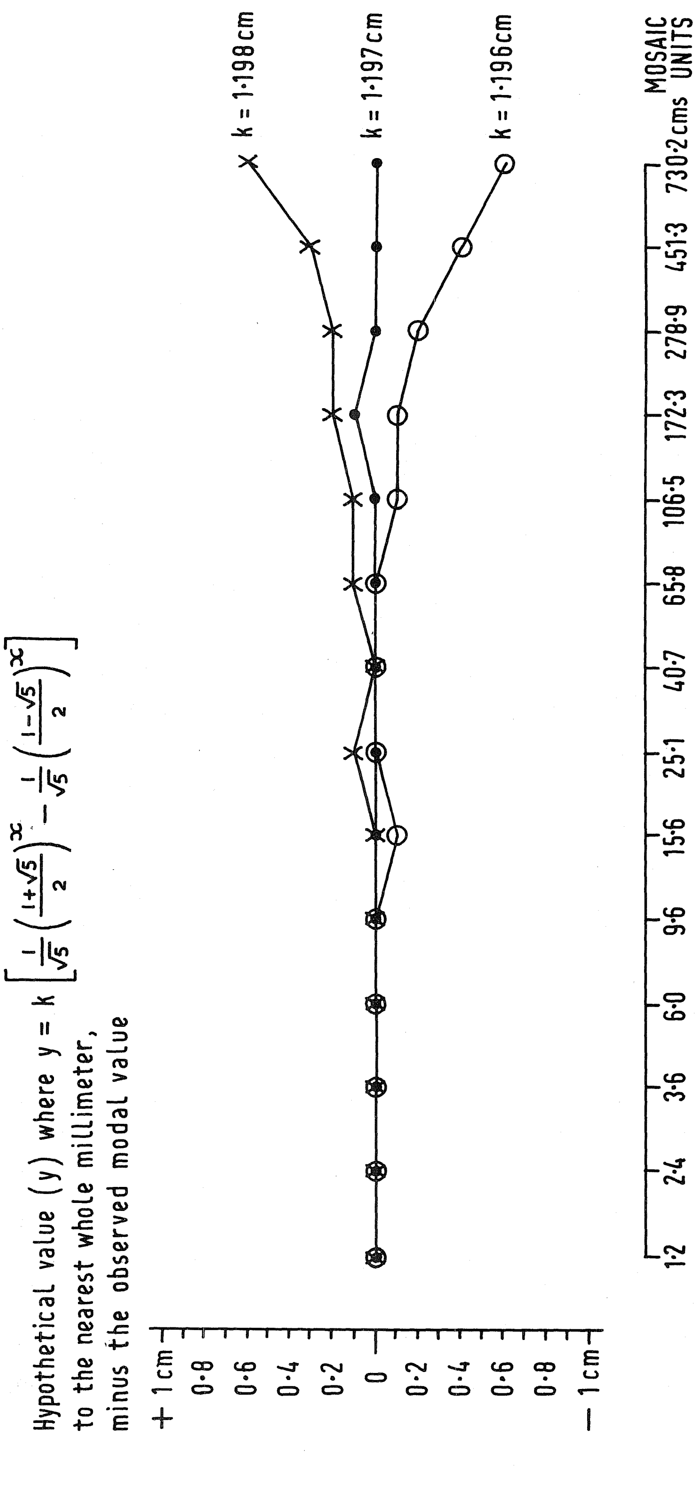

Hypothetical value (y) where $y = k\left[\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{x} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{x}\right]$
to the nearest whole millimeter,
minus the observed modal value
+ 1 cm
0·8
0·6
0·4
0·2
0
0·2
0·4
0·6
0·8
− 1 cm
k = 1·198 cm
k = 1·197 cm
k = 1·196 cm
1·2
2·4
3·6
6·0
9·6
15·6
25·1
40·7
65·8
106·5
172·3
278·9
451·3
730·2 cms
MOSAIC UNITS

Figure 6

2 x 25.137 = 50.274 ≡ 50.3 cm, matching the observed value. A minor peak occurs at 75.6 cm, but 3 x 25.1 = 75.3, and 3 x 25.137 = 75.4. Here, the observation slightly exceeds both theoretical values.

Small peaks at multiples of unit 9.6 cm occur (Fig. 2). However (Fig. 7) the observed values progressively increase beyond the theoretical values, taking the 5th mosaic unit as 9.576 cm. Although the observed values are fitted (Fig. 7) by values based on the assumption that this unit measures exactly 9.6 cm, this does not necessarily mean that mosaic units were based on 1.2 cm rather than 1.197 cm. If values that are multiples of others were measured out by repeating measurement of the basic unit the desired number of times, greater error would tend to accompany greater multiples. This error would tend mostly to add to the intended length (the observed condition) if rulers were butted end to end to achieve it. Providing rulers are not displaced too much sideways, and as they are unlikely to be compressable, errors will tend to add to the intended value rather than reduce it.

Lack of symmetry of some peaks might be expected if the first unit was 1.197 cm long. For example, in the case of the 6th mosaic unit, the observed peak has a modal value of 15.6 cm. Its theoretical value based on 1.197 cm is 15.561 cm. Although this lies within the ±0.05 cm range of the observed modal mid-interval, it lies very much to the left of the mid-interval (-0.04 cm). We might expect that the observations would form an asymmetrical peak, more values occurring in the left-hand half of the peak. I detect (Fig. 8) no clear tendency for this effect in the present data.

WHY WERE MOSAIC UNITS USED?

The ancient names for some everyday units of length which refer to finger, knuckle, palm, handspan, handlength, etc., suggest that people once actually used their limbs to measure things. Some tradesmen still measure out yards by the tip of their nose to their sideways stretched fist. According to Vitruvius [5], "Besides, the ancients took from the members of the human body the proportional (?) dimensions needed in all constructions, finger, palm, foot, cubit." Some [6] trace this back to Plato in the <u>Theatus</u>, "Man is the measure of all things."

If mosaicists used their fingers, hands, etc., for measuring out their patterns, would this give rise to the observed situation? Whilst it might cause

Observed modal value	Number of cases	Multiple of mosaic unit 9·6 cm	Hypothetical value based on 1·197 cm	Hypothetical value based on 1·2 cm
9·6 cm	15231	1	9·6 cm	9·6 cm
19·2	265	2	19·2	19·2
28·8	97	3	28·7	28·8
38·5	31	4	38·3	38·4
48·0	99	5	47·9	48·0
57·6	36	6	57·5	57·6
67·3	30	7	67·0	67·2
76·8	46	8	76·6	76·8
none	0	9	86·2	86·4
96·0	76	10	95·8	96·0
none	0	>10	—	—

Figure 7

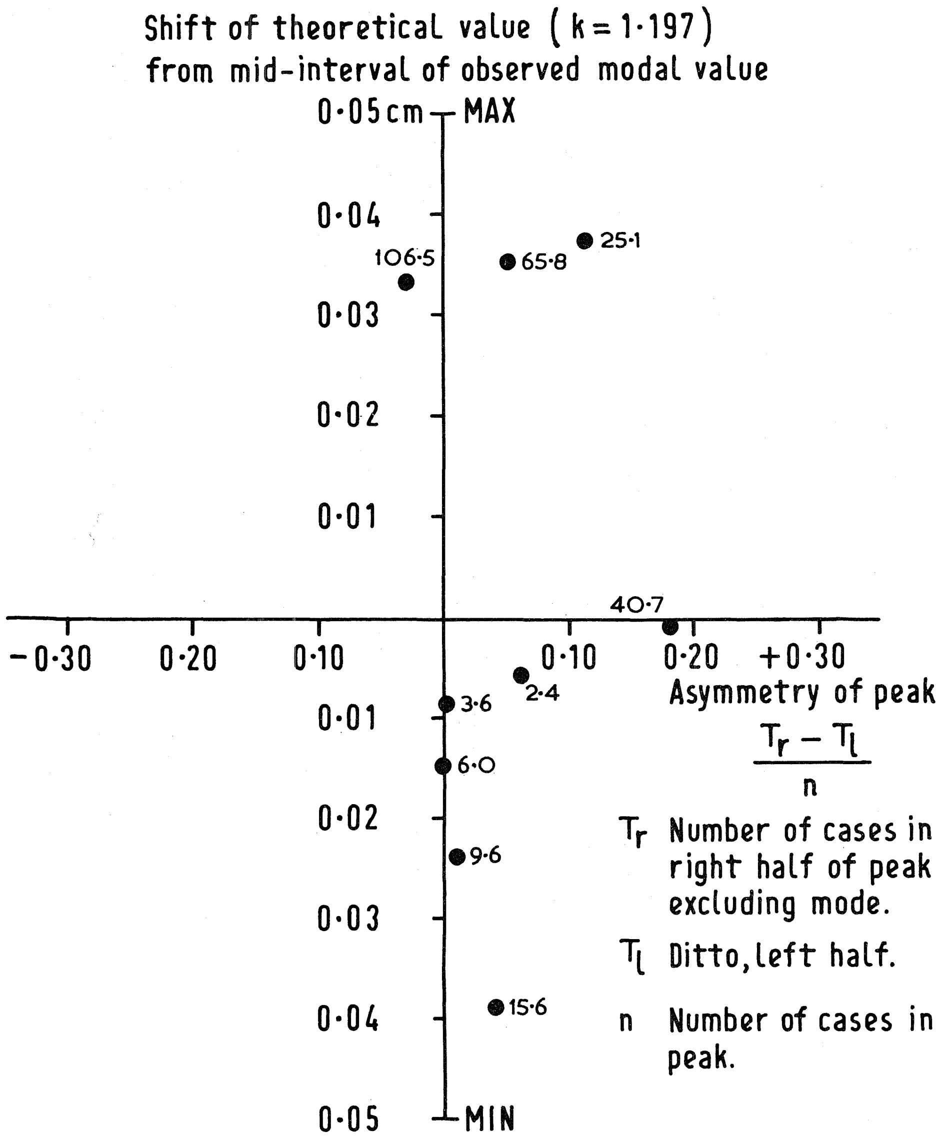

Shift of theoretical value (k = 1·197)
from mid-interval of observed modal value
0·05cm
MAX
0·04
0·03
0·02
0·01
106·5
65·8
25·1
40·7
−0·30
0·20
0·10
0·10
0·20
+0·30
2·4
3·6
6·0
9·6
15·6
0·01
0·02
0·03
0·04
0·05
MIN
Asymmetry of peak
$\frac{T_r - T_l}{n}$
T_r Number of cases in right half of peak excluding mode.
T_l Ditto, left half.
n Number of cases in peak.

Figure 8

patterns to fall into size groups, the variation between, say, the handlength on different people is far [4] in excess of the range of distances contributing to a typical mosaic pattern size.

However, the variation would be much reduced if, instead of each mosaicist using his own hand length, if each used a ruler calibrated from one single man. Slight support for this exists in that limb dimensions do roughly fit mosaic units. For example, my own approximate dimensions are as follows: First joint of index finger 2.5, first joint of thumb 3.5, thumb length 6.0, length of index finger 9.5, the "spithama" (tip of index finger to tip of thumb spread wide) 15.5, hand span 21.5, foot 25.0, "inner cubit" (tip of index finger to inner bend of bent elbow) 40.0, arm length with fist clenched 66.0 cms, respectively.

Ancient units of length with anthropormorphic names, however, measure distances less coincident with mosaic units than this. For example, a typical Greek standard span is about 23.0 cm. Roman and Greek standard cubits mostly lie between 42.35 cm [7] and 46.000 cm [8], and some Hebrew cubits lie outside this range. Egyptian and Sumerian cubits are mostly longer; there is even a Chino-Sumerian cubit of 74.40 cm!

In discussing why the ancients chose to so consistently make their mosaic patterns one or other of the set of mosaic units, it might be useful to express mosaic units in ancient units of length rather than in a modern system. Comparing 6646 values derived as whole multiples and likely fractions of so ancient units of length possibly pertinent to the mosaic craft, I find very few mosaic units are equal to a whole multiple (or multiple plus likely fraction) of a known standard unit. The only single standard unit which yields more than about two mosaic units appears to be the Greek finger of 1.92 cm [9], and this only fits five of the eleven mosaic units (Fig. 9).

However, expressing the mosaic units (ignoring 21.6 cm) in terms of this Greek finger yields (to the nearest whole number) integers which are the actual Fibonacci numbers up to unit 65.8 cm (Fig. 10). This could be significant, for expressing mosaic units in modern units produces integers (Fig. 10) lacking this property. Neither (Fig. 10) does the Roman digit from most Roman feet fit so well.

Measuring in centimeters does, however, bring out the relationship 10 x unit 3.6 cm = 6 x unit 6.0 cm. This relation could be significant, for sexagesimal relations are common in ancient metrology (some effects of which are

Mosaic unit	Multiple of Greek finger	Length yielded by this multiple
2·4cm	1 ¼	2·40cm
9·6	5	9·60
21·6	11 ¼	21·60
25·1	13	24·96
40·7	21 ¼	40·80

Figure 9

MOSAIC UNITS EXPRESSED IN VARIOUS UNITS OF MEASURE

Cms.	nearest integer	Present English inch	integer	Greek finger of 1·92cm	integer	Roman digit of 1·89cm	integer
1·2	1 ●	0·47	0	0·1	0	0·1	0
2·4	2 ●	0·94	1 ●	1·3	1 ●	1·3	1 ●
3·6	4	1·41	1 ●	1·8	2 ●	1·9	2 ●
6·0	6	2·36	2 ●	3·1	3 ●	3·2	3 ●
9·6	10	3·77	4 ●	5·0	5 ●	5·1	5 ●
15·6	16	6·12	6 ●	8·1	8 ●	8·3	8 ●
(21·6)							
25·1	25	9·89	10	13·1	13 ●	13·3	13 ●
40·7	41	16·01	16	21·2	21 ●	21·5	22
65·8	66	25·90	26	34·3	34 ●	34·8	35
106·5	107	41·92	42	55·5	56	56·3	56

Fibonacci number marked ●

Figure 10

still with us, e.g., division of 1° into 60 min. of arc). However, that $1.197 = \sqrt[10]{6}$ can be dismissed as significant, for it is a result of using centimeters. Also the otherwise attractive relation that a right angle triangle of sides 10 and 6 has a hypotenuse of just under 12.

Translating Vitruvius' next remarks, we learn that, "The Ancients grouped these body dimensions into the perfect number called teleon. They decided on the number ten as perfect···. But mathematicians, in disagreement, say the number six is perfect···. Later they realized both six and ten are perfect, and they put them together, making the most perfect number sixteen···." As it happens, 16 x unit 6.0 cm = 10 x unit 9.6 cm.

Dr. George Ledin, Jr. has extracted [10] Fibonacci numbers from mosaic units by dividing the observed values in centimeters by 1.19, obtaining integer 18 from unit 21.6 cm. He has found [10] that the unit 21.6 cm, which is "odd" in the sense that all the others are directly related to Fibonacci numbers, can itself be related to the Fibonacci series, for 18 is a term in the Lucas series. Although multiplying 1.197 cm by 18 gives 21.5, not 21.6 cm as observed, adding unit 5.985 cm to unit 15.561 gives the same result.

Ledin [10] draws attention to the connection: mosaic units $\rightarrow$ Fibonacci numbers $\rightarrow$ the so-called "golden section." Firm evidence that the ancients knew, and regarded as special, the "golden" ratio 1:1.618··· is provided by Euclid's Elements Book 6, Definition 3 and Proposition 30. But did the ancients know the Fibonacci series? D'Arcy Thompson has said [11], "...there is no account of it, nor the least allusion to it, in all the history of Greek mathematics...," but also [11], "It is quite inconceivable that the Greeks should have been unacquainted with so simple, so interesting, and so important a series; so clearly connected with, so similar in its properties to, that table of side and diagonal numbers which they knew familiarly."

If the ancients did use mosaic units because of their connection with 1:1.618, it seems to imply that they knew the Fibonacci series. If this could be shown to be their reason, we would apparently have unique evidence of knowledge of the Fibonacci numbers (F_x) before Leonardo of Pisa. It would also mean that the knowledge

$$\lim_{x \to \infty} \frac{F_{x+1}}{F_x} = 1.618 \cdots$$

existed before Kepler, who is apparently [12] regarded as the first to know it. Simply because written record of the series 1, 1, 2, 3, 5, 8, ··· has not come down to us from the Greeks of course does not mean that they did not know it. A dramatic example is the recent discovery [13] of the unexpected ancient Greek computing mechanism, complete with dials and gearing, to which no known allusion had reached us either.

The ratio between successive pairs of mosaic units greater than the pair 6.0:9.6 is close to 1:1.616 (ignoring 21.6 cm). The ratio 6.0:9.6 is 1:1.600. The ratios for the smaller units are 1:1.6, 1:1.5, and 1:2. Had the mosaicists invoked the ratio 1:1.618 (without the Fibonacci series) we might expect their units measuring less than the pair 9.6:15.6 cm to exhibit the 1:1.618 ratio. Their smaller units would then measure 5.993, 3.667, 2.666, 1.648 cm, respectively. As such a series can be extrapolated indefinitely, we might expect another unit at 1.002 cm, another at 0.619 cm, and so on. However, the observed frequency distribution of pattern sizes does not support this idea.

Alternatively, if the ancients simply wanted units each the sum of the preceding two, and wanted this in order to invoke the ratio 1:1.618, there is no need for the units to be the lengths I call mosaic units. Consider the general series $U_{n+2} = U_{n+1} + U_n$.

$$U_{n+2} - U_{n+1} - U_n = 0.$$

Put $U_n = At^n$ where A and t are any two constants. Then

$$At^{n+2} - At^{n+1} - At^n = 0.$$

$$t^2 - t - 1 = 0 \qquad (t \neq 0).$$

So

$$t = \frac{1 \pm \sqrt{1 - (-4)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Therefore,

$$U_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n$$

where a and b are determined by the initial conditions; the values of the first two terms.

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{a\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} + b\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n} .$$

Since

$$0 < \left|\frac{1-\sqrt{5}}{2}\right| < 1 ,$$

then

$$\lim_{n\to\infty} \left(\frac{1-\sqrt{5}}{2}\right)^n = 0 .$$

Thus, by simplification,

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1+\sqrt{5}}{2} = 1.618\cdots .$$

Thus, very many different sets of units could have been used in mosaic construction, yet all possessing a common property of the ratio between successive units approaching 1:1.618.

As it is, the consistent use of a particular set of absolute lengths (mosaic units) apparently throughout the classical world from about 400 B.C. to 530 A.D. (the limits of my observations) could suggest some particular need. Could it be a practical matter?

Pouring cleaned ancient mosaic stones (given to me by the Italian government and lent by the British Museum for the purpose) into a machine I made to pack stones in adjacent rows in the manner of a mosaic, a strip of mosaic was formed with stones in random sequence. Inspection of successively formed machine made mosaics revealed [14] a tendency for stones in adjacent rows to periodically align transversely across the rows (Fig. 11).

Classical floor mosaicists apparently [2], [15], [16] normally laid their stones without specially selecting them for size, or shaping them to suit. It is possible that they found this alignment phenomenon for themselves, and from then on took advantage of it by making their pattern sizes which coincide with the tendency for alignments, thus causing less erratic rims to their patterns (and for no extra effort).

That the ancients may have noticed alignments is made more probable by the existence of alignments [14] in unpatterned ancient mosaics (Fig. 11). Alternatively, they could simply have noticed that their patterns often turned out most regular when made certain sizes, and from then on intentionally made patterns these sizes, without noticing the alignments as such.

Measurements I made of the distances between alignments in intact unpatterned ancient mosaics largely coincide [14] with the typical intervals between alignments in my machine-made mosaics. These, in turn, agree [14] with the typical pattern sizes — mosaic units.

I find many mosaic patterns lie in sequence such that their dimensions lie in the same sequence as the units would lie if marked in ascending order on a ruler (Fig. 12). It seems likely that mosaicists might notice, even if they did not know beforehand, that the distance between two adjacent calibrations on the ruler also measures one of the units in the same set. From this, it would seem a short step to perceiving that each unit (except 21.6 cm) is the sum of the preceding two. If they regarded the first unit as unity, and ignored unit 21.6 cm, they would have arrived at the Fibonacci series empirically.

The liklihood of the Fibonacci properties of mosaic units being originally intentional is linked with whether it is correct to regard unit 21.6 cm as secondary, thus leaving the units possessing Fibonacci properties. In the sense that 21.6 cm can be expressed in terms of the other units (6.0 + 15.6 cm), 21.6 cm does seem secondary.

On the ruler (Fig. 13A) mosaic unit 21.6 cm is symmetrical with 9.6 cm about unit 15.6 cm, in the same way that 25.1 cm is symmetrical about 15.6 cm with 6.0 cm. The most frequent pattern sizes I have found that are not single

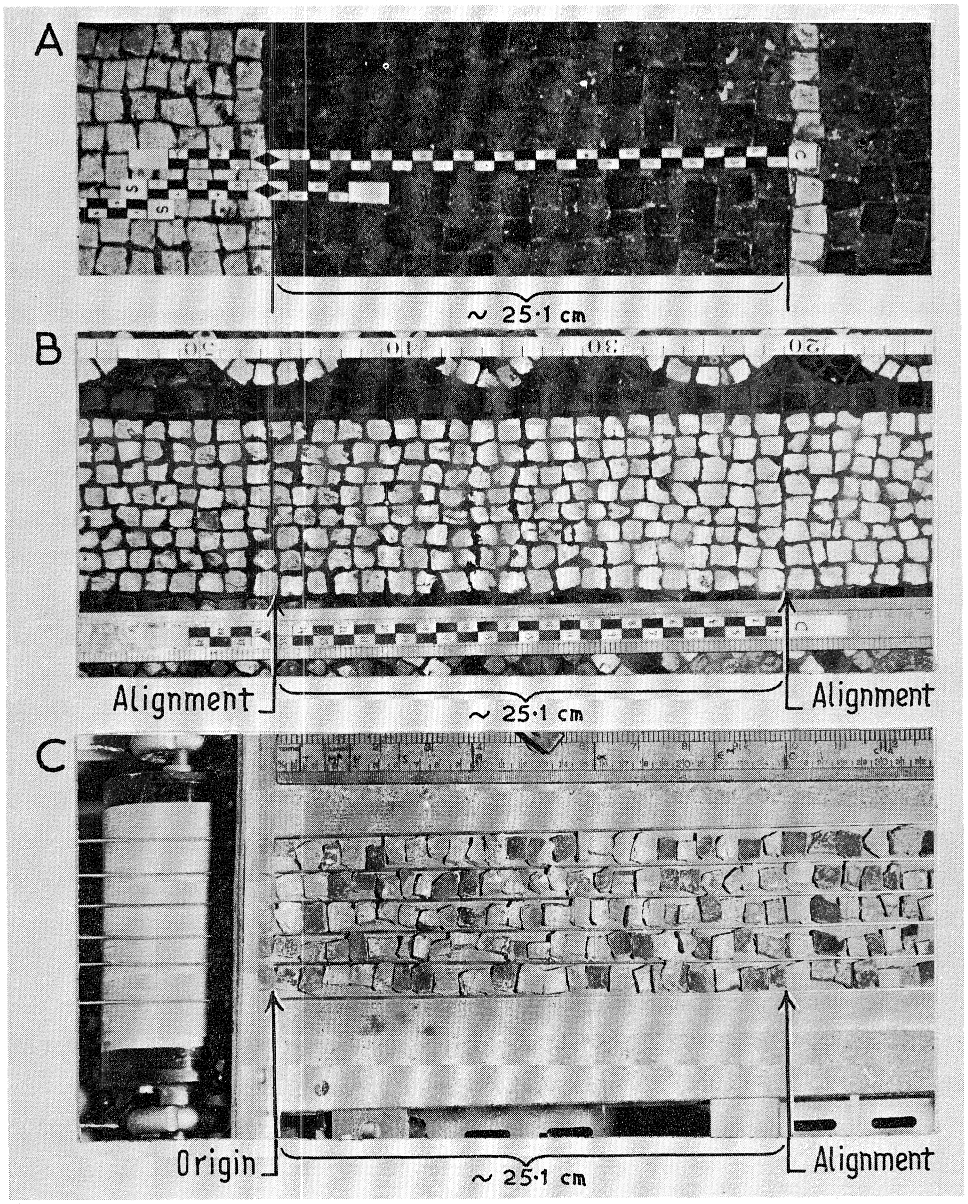


Figure 11

A. Typical ancient mosaic pattern (Markets of Trajan, Rome). Pattern dimension agrees with mosaic unit 25.1 cm.

B. Typical pair of alignments in an intact Roman mosaic (Aldborough, England). Alignment interval agrees with mosaic unit 25.1 cm.

C. Typical alignment which occurred as a packing pehnomenon among loose ancient mosaic stones packed in rows by machine. Distance of alignment from starting place of aggregation agrees with mosaic unit 25.1 cm.

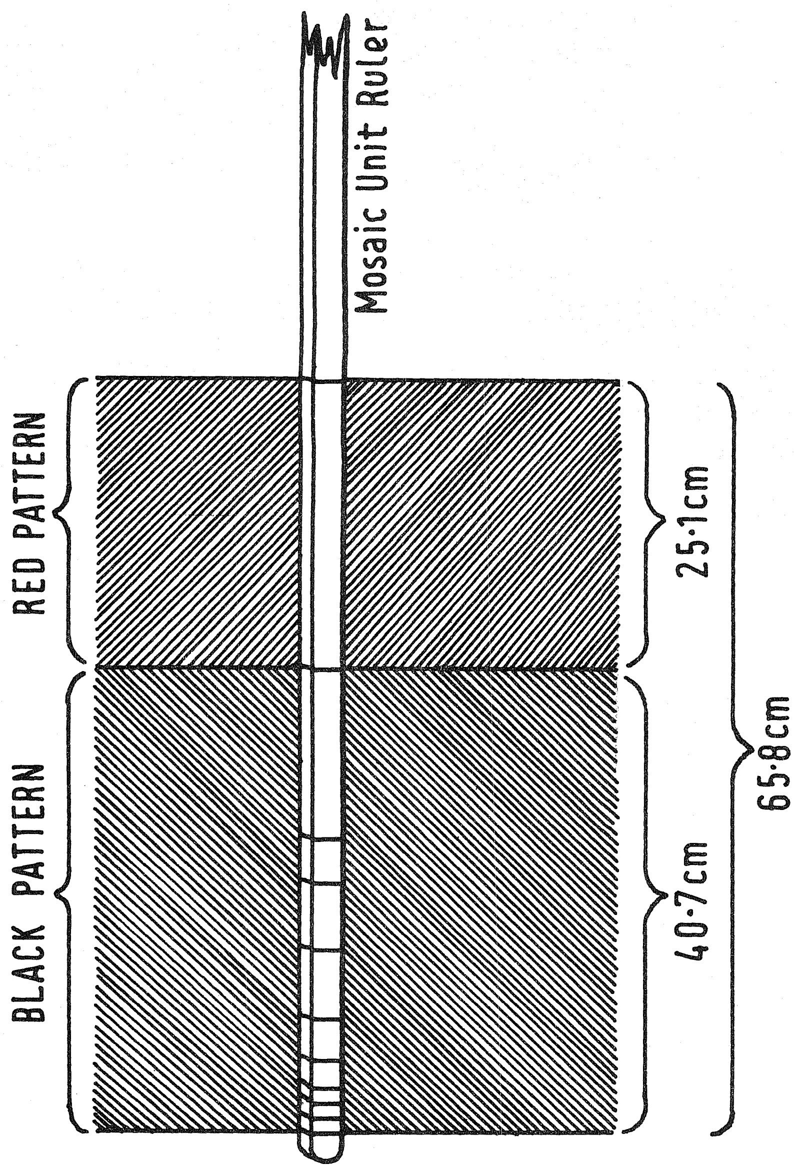

Mosaic Unit Ruler
RED PATTERN
BLACK PATTERN
25·1cm
40·7cm
65·8cm

Figure 12

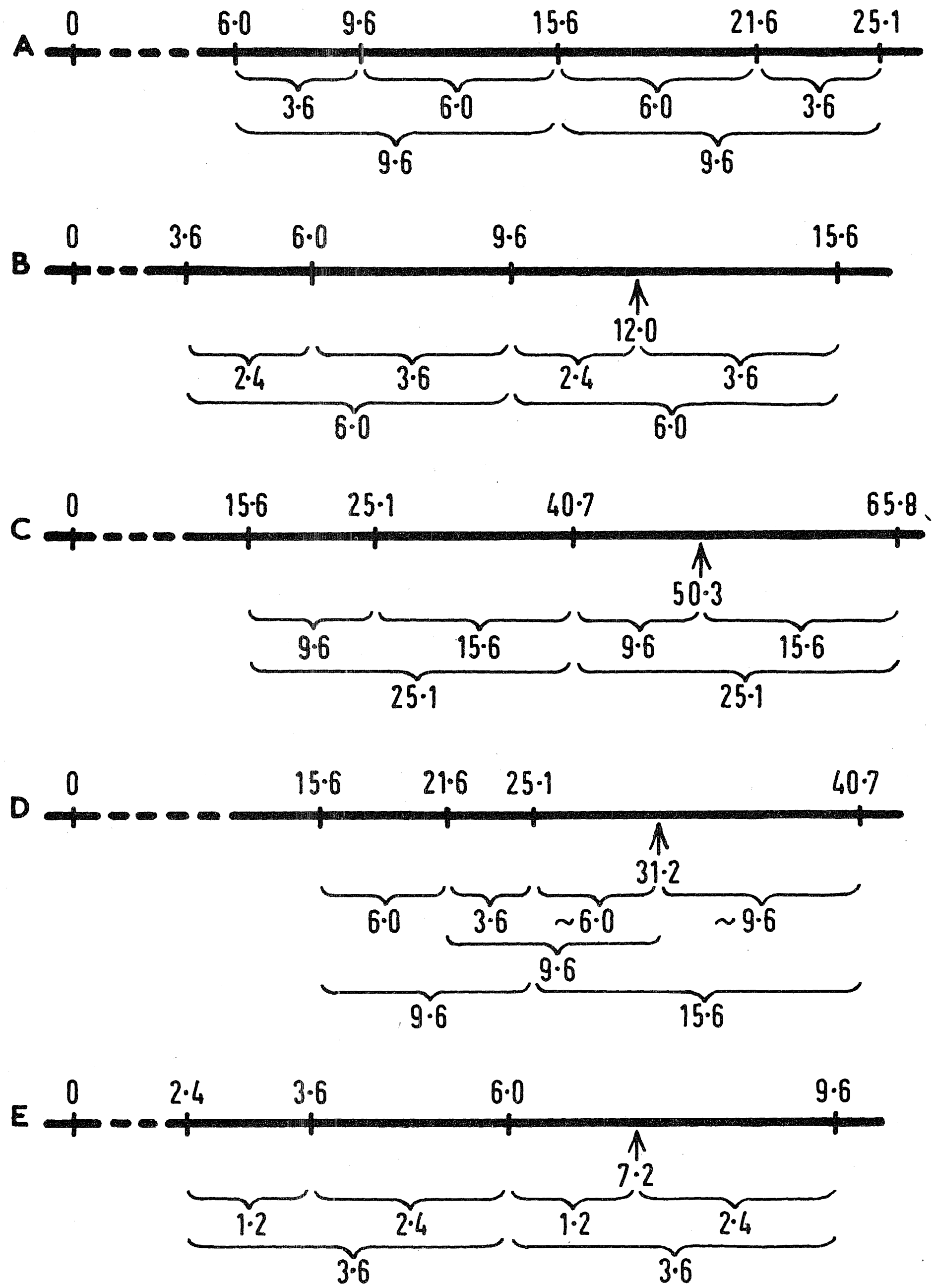

A
0
6·0
9·6
15·6
21·6
25·1
3·6
6·0
6·0
3·6
9·6
9·6
B
0
3·6
6·0
9·6
15·6
12·0
2·4
3·6
2·4
3·6
6·0
6·0
C
0
15·6
25·1
40·7
65·8
50·3
9·6
15·6
9·6
15·6
25·1
25·1
D
0
15·6
21·6
25·1
40·7
31·2
6·0
3·6
~6·0
~9·6
9·6
9·6
15·6
E
0
2·4
3·6
6·0
9·6
7·2
1·2
2·4
1·2
2·4
3·6
3·6

Figure 13

mosaic unit dimensions are 12.0, 50.3, 31.2, and 7.2 cm, respectively. They all possess similar symmetrical properties (Fig. 13, B-E).

A $\sqrt{2}$ relationship is known [17] between the Royal cubit of Herodotus and the Egyptian remen. A square of side 21.6 cm has a diagonal of 30.55 cm (the modern English foot is 30.48 cm). Many ancient standard feet are known, most varying between the 29.3 cm Roman foot [18] to a foot used in Roman Europe and elsewhere of 33.5 cm [7]. A typical value to the nearest whole millimeter is 30.8 cm for the Roman foot and 30.6 cm for the Greek. The latter fits this diagonal of 30.6 cm. That again Greek measure fits better than Roman would fit in with the mosaic craft being passed from the Greeks to the Romans.

However, the value of 21.6 cm occurred, by natural causes, and about as frequently [14] as 6.0, 9.6, 15.6, and 25.1 cm in the form of the interval between alignments in both intact ancient mosaics and in my experimentally produced mosaics. From this point of view it seems as basic as the other values.

That the pattern sizes tend to be the same few lengths for so long a period (1000 years) and over such a geographic extent, seems to me to indicate a practical reason rather than common subscription to some aesthetic. While aesthetic principles were apparently invoked in temple construction, it is less likely that they would be "debased" to the level of crude floor covering. The majority of mosaics in the present study are very humble, rife with imperfections and even errors. Even now, when they have the extra quality of "ancient" to recommend them, many are regarded as not worth bothering about, allowed to fall to pieces, or are openly permitted to be damaged.

In contrast, original mistakes were not normally allowed to remain in the kind of work to which aesthetic principles were applied; witness the remarkable perfection of Greek temples. According to Vitruvius [5], "... the ancients have, in their works, determined that each part be an aliquot (?) part of the total plan... especially in temples, wherein faults as well as beauties will last for all time." The latter is connected with ancient concern over the area contained by a ground plan. For example, Plutarch [19] calls the Parthanon hecatompedon (i.e., the "hundred-footed"). The word <u>templum</u> (a temple) originally meant [20] a space that could be, or is, enclosed. Possibly this concern with the area of a floor came from concern over the area (also called "templum") within which the flight of birds was watched when looking for omens to guide decisions. Fixing the boundary of this area would be crucial to a believer, it must contain any relevant flight that might occur, but must not be too big so as to make proper watching impossible. The notion of an "ideal" also presents itself.

Standardization of building material dimensions [21] is another factor leading to floor areas being defined in whole numbers of units. The latter is used by metrologists in deducing lengths of some ancient standard units. Flinders Petrie found [7] an average of only about 5mm "original" error per meter in ancient measured lengths.

That so many floor mosaics do not fit their floor area suggests lack of relationship between the units used in their construction and the normal standard units fixing the floor area. That "special" units should be used for mosaics would fit the contrast [22] at one time between the cubit used for everyday life and the special cubit reserved for building.

Given a stock of mosaic stones of very high constancy of size (a most unusual condition for ancient mosaic stones) and they are of side length equal to the smallest mosaic unit, one can construct the other mosaic units as follows (Fig. 14A). Set down one stone. Its side gives a length of 1.2 cm. Place another next to it. We have a new dimension of 2.4 cm. Adding the latter stone was equivalent to squaring the original length, for taking the side length of the first stone as unity. $1^2 = 1$, so we added one stone, thereby obtaining the new length of two stones. Square this new dimension (2^2) adding four stones in a square. We obtain a new length of $2 + 1$ stones $= 3.6$ cm. Square this new dimension, and 6.0 emerges. Continue this process, and all the mosaic units (except 21.6 cm) are formed. By simply adding stones until a square is formed on each new dimension, there is no need even to count stones.

However, if, instead of moving around the starting stone, Dr. Michael Whippman of Pennsylvania University has pointed out to me that moving from side to side of the growing diagonal of the overall construction, as in his figure (Fig. 14B), the value 21.6 cm can also be obtained. He suggests that the mosaicists' ruler could have been of this "flat plate" form.

Dr. Wayne Cole of Abbott Laboratories, Illinois, has suggested to me that if the mosaicists used a tool of this form, it need not be bigger than unit 40.7 cm. When the need arises, unit 65.8 cm can be obtained by using first

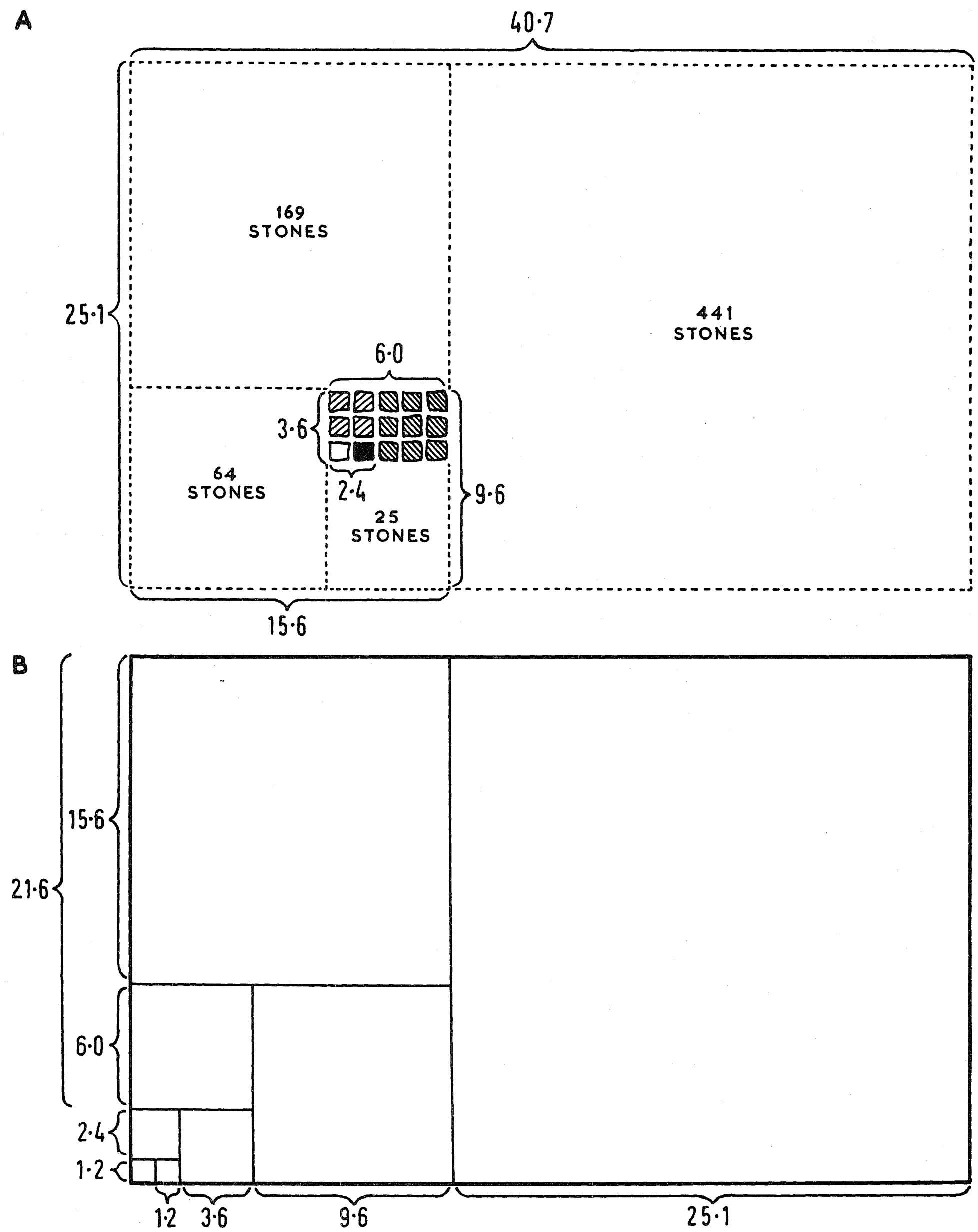

A
40·7
169
STONES
441
STONES
25·1
6·0
3·6
64
STONES
2·4
25
STONES
9·6
15·6
B
15·6
21·6
6·0
2·4
1·2
1·2
3·6
9·6
25·1

Figure 14

one edge of the tool and then the other (40.7 + 25.1 = 65.8 cm). If unit 106.5 cm is required, using the long edge twice and then the other fulfills this (40.7 + 40.7 + 25.1 = 106.5 cm). He also suggests that such a tool could double as a square.

Ancient rulers, often calibrated in inadequately examined units, have come down to us. It is possible that an original mosaicists' ruler may still exist. If one does come to light, it might indicate how mosaic units arose. For example, many ancient rulers are square sectioned sticks with saw cuts for calibrations. On some, the cuts run around all four faces of the ruler for the prime units and only on one or two faces for other values. If the unit 21.6 cm was differentiated in this way, its secondary nature would seem established, emphasizing the Fibonacci properties of the others.

In 1632, A. Bosio [23] gave an engraving showing a Roman tomb on which is depicted two kinds of dividers, a peg and line, a level, a chisel, a punch, a sharp bladed hammer, a square, and a ruler (Fig. 15). The calibrations on rulers shown on other monuments have been found to be accurate [24]. In the engraving, the two smallest divisions on the ruler, each marked by a dot, are equal, and the interval marked "R" is equal to the unmarked interval at the extreme right. The interval marked "dot A dot" is equal to interval "R" plus the two intervals marked with dots. All these intervals can be constructed accordingly, providing we can fix the position for the second calibration from the right, and the size of the interval marked "dot."

Taking the unmarked interval as unity, measurement shows that the interval marked "dot" is 0.618 long. Thus "dot" corresponds to $1/\tau$. "R" corresponds to $1 + (2/\tau)$, where τ is 1:1.618.

If the engraving can be relied on, this is a case where the ancients apparently intentionally used units with 1:1.618 inter-relationships. The proportions

$$\left(1 + \frac{2}{\tau}\right) : 1 : \frac{1}{\tau} \quad ,$$

can be fitted by 21.6:15:6:6.0 cm. Thus the calibrations on Bosio's ruler could be these three mosaic units. I hope to find whether this tomb-face still exists, and whether this ruler does match mosaic units.

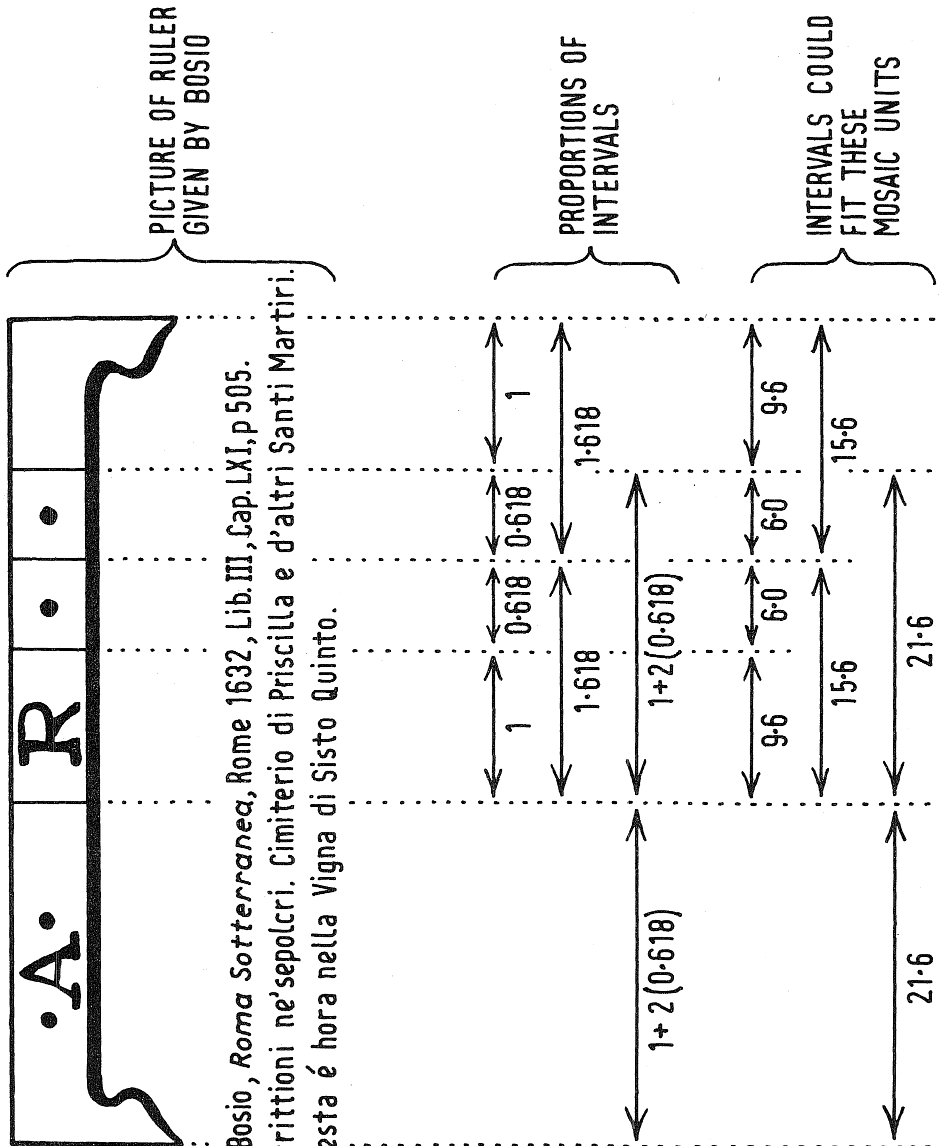

PICTURE OF RULER GIVEN BY BOSIO
PROPORTIONS OF INTERVALS
INTERVALS COULD FIT THESE MOSAIC UNITS
·A· R · ·
A. Bosio, *Roma Sotterranea*, Rome 1632, Lib. III, Cap. LXI, p 505.
Iscrittioni ne'sepolcri. Cimiterio di Priscilla e d'altri Santi Martiri.
Questa é hora nella Vigna di Sisto Quinto.
1
0·618
0·618
1
1·618
1·618
1+2(0·618)
1+2(0·618)
9·6
6·0
6·0
9·6
15·6
15·6
21·6
21·6

Figure 15

The origin of the craft of mosaic is unknown to us, so it is hard to know what ideas and knowledge were current during the development of mosaic techniques. From the point of view of patterns sizes, I find the oldest (400 B.C.) known Greek mosaics exhibit the same dimensions as were apparently customarily used throughout the Greek and Roman world thereafter. The floor mosaic patterns at Til Barsib and Arslan Tash (c. 900 B.C.) in Syria, from the information available, appear to be also mosaic unit sizes. There is a distinct lack of primative mosaics in which we might see the mosaic unit phenomenon gradually developing. This could suggest that there are mosaics older than the earliest we at present know, but, if they were of the type, where pebbles are simply placed in earth, they have probably perished.

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[Continued on page 334.]