Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania, 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-172 Proposed by David Englund, Rockford College, Rockford, Illinois.

Prove or disprove the "identity,"

\[ F_{kn} = \sum_{t=1}^{\left[ \frac{k+1}{2} \right]} (-1)^{(n+1)(t+1)} \binom{k-t}{t} L_n^{k-2t+1} \]

where \( F_n \) and \( L_n \) denote the \( n \)th Fibonacci and Lucas numbers, respectively, and \( [x] \) denotes the greatest integer function.

H-173 Proposed by George Ledin, Jr., Institute of Chemical Biology, University of San Francisco, San Francisco, California.

Solve the Diophantine equation,

\[ x^2 + y^2 + 1 = 3xy \]


Let \( k \) be any non-zero integer and \( \{S_n\}_{n=1}^{\infty} \) be the sequence defined by \( S_n = nk \).

Define the Burn's Function, \( B(k) \), as follows: \( B(k) \) is the minimal value of \( n \) for which each of the ten digits, 0, 1, \ldots, 9, have occurred
in at least one $S_m$ where $1 \leq m \leq n$. For example, $B(1) = 10$, $B(2) = 45$. Does $B(k)$ exist for all $k$? If so, find an effective formula or algorithm for calculating it.

SOLUTIONS

OLDIES BUT GOODIES

The following problems are still lacking solutions:

1. H-22
2. H-46
3. H-74
4. H-86
5. H-94
6. H-104
7. H-108
8. H-115
9. H-125
10. H-23
11. H-60
12. H-76
13. H-87
14. H-100
15. H-105
16. H-110
17. H-116
18. H-127
19. H-40
20. H-61
21. H-77
22. H-90
23. H-102
24. H-106
25. H-113
26. H-118
27. H-130
28. H-43
29. H-73
30. H-84
31. H-91
32. H-93
33. H-107
34. H-114
35. H-122

GENERATING FUNCTIONS

H-144 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A. Put

$$[(1 - x)(1 - y)(1 - ax)(1 - by)]^{-1} = \sum_{m,n=0}^{\infty} A_{m,n} x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} A_{n,n} x^n = \frac{1 - abx^2}{(1 - x)(1 - ax)(1 - bx)(1 - abx)}. $$

B. Put

$$(1 - x)^{-1}(1 - y)^{-1}(1 - axy)^{-\lambda} = \sum_{m,n=0}^{\infty} B_{m,n} x^m y^n .$$

Show that
\[ \sum_{n=0}^{\infty} B_{n,n} x^n = (1 - x)^{-1} (1 - ax)^{-\lambda}. \]

*Solution by the Proposer.*

Solution, A. We have

\[ A_{m,n} = \sum_{i=0}^{m} \sum_{j=0}^{n} a^i b^j = \frac{(1 - a^{m+1})(1 - b^{n+1})}{(1 - a)(1 - b)}, \]

so that

\[ \sum_{n=0}^{\infty} A_{m,n} x^n = \sum_{n=0}^{\infty} \frac{(1 - a^{n+1})(1 - b^{n+1})}{(1 - a)(1 - b)} x^n \]

\[ = \frac{1}{(1 - a)(1 - b)} \left\{ \frac{1}{1 - x} - \frac{a}{1 - ax} - \frac{b}{1 - bx} + \frac{ab}{1 - abx} \right\} \]

\[ = \frac{1 - ax}{1 - x} \left\{ \frac{1}{1 - ax(1 - ax)} - \frac{b}{(1 - bx)(1 - abx)} \right\} \]

\[ = \frac{1 - abx^2}{(1 - x)(1 - ax)(1 - bx)(1 - abx)}. \]

Solution, B. We have

\[ (1 - x)^{-1} (1 - y)^{-1} (1 - axy)^{-\lambda} \]

\[ = \sum_{r,s,t=0}^{\infty} \frac{(\lambda)_t}{t!} a^{r+t} x^s y^t, \]

where

\[ (\lambda)_t = (\lambda - 1)(\lambda - 2) \cdots (\lambda - t + 1) \quad (t \geq 1) \quad \text{and} \quad (\lambda)_0 = 1, \]
so that

\[ B_{m,n} = \sum_{t=0}^{\min(m,n)} \frac{\lambda_t}{t!} a^t. \]

Hence

\[
\sum_{n=0}^{\infty} B_{n,n} x^n = \sum_{n=0}^{\infty} x^n \sum_{t=0}^{\infty} \frac{\lambda_t}{t!} a^t
\]

\[ = \sum_{t=0}^{\infty} \frac{\lambda_t}{t!} (ax)^t \sum_{n=0}^{\infty} x^n
\]

\[ = (1 - x)^{-1} (1 - ax)^{-\lambda}. \]

Also solved by M. Yoder and D. Jaiswal.

FACTOR ANALYSIS

H-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

If

\[ n = e_1 e_2 \cdots e_r \]

is the canonical factorization of \( n \), let \( \lambda(n) = e_1 + \cdots + e_r \). Show that \( \lambda(n) \leq \lambda(F_n) + 1 \) for all \( n \), where \( F_n \) is the \( n \)th Fibonacci number.

Solution by the Proposer.

Clearly, \( \lambda(mn) = \lambda(m) + \lambda(n) \), and if \( m \mid n \) then \( \lambda(m) \leq \lambda(n) \). Also, \( 1 = \lambda(p) \leq \lambda(F_p) \) for any prime \( p \). We show by induction that \( \lambda(p^k) \leq \lambda(F_{p^k}) \) for all \( k \), except when \( p = k = 2 \), when \( \lambda(4) = \lambda(F_4) + 1 \). The cases when \( p^k \leq 12 \) are checked directly. Assume the result is true for \( p^{k-1} \). Then since \( p^k > 12 \), by Carmichael's theorem ("On the Numerical
Factors of the Arithmetical Forms $a^n \pm b^n$, "Annals of Math. (2nd Ser.), 15, pp. 30-70, Theorem XXIII) there is a prime dividing $F_{p^k}$ not dividing $F_{p^k-1}$.

Then since $F_{p^k-1} | F_{p^k}$, we have

$$\lambda(F_{p^k}) \geq 1 + \lambda(F_{p^k-1}) \geq k,$$

completing the induction. Hence $\lambda(p^k) \leq \lambda(F_{p^k})$ except when $p = k = 2$.

In the factorization

$$n = p_1^{e_1} \cdots p_r^{e_r},$$

we can assume $p_1 = 2$, and $e_1 = 0$ if necessary. Then

$$F_{e_1^{p_1}} \cdots F_{e_r^{p_r}}$$

are pairwise relatively prime since $p_1^{e_1}, \ldots, p_r^{e_r}$ are, and since $F_{e_1}$ divides $F_n$ for each $i$, so their product

$$F_{e_1} \cdots F_{e_r} | F_n.$$

Hence,

$$\lambda(F_n) \geq \lambda(F_{e_1} \cdots F_{e_r}) = \lambda(F_{e_1}) + \cdots + \lambda(F_{e_r}) \geq \lambda(n) - 1,$$

which completes the proof.

*Also solved by M. Yoder.*

**CONVERGING FRACTIONS**

*H-147 Proposed by George Ledin, Jr., University of San Francisco, San Francisco, California.*

Find the following limits. $F_k$ is the $k$th Fibonacci number, $L_k$ is the $k$th Lucas number, $\pi = 3.14159\cdots$, $\alpha = (1 + \sqrt{5})/2 = 1.61803\cdots$, $m = 1, 2, 3, \cdots$. 
Solution by David Zeitlin, Minneapolis, Minnesota.

EDITORIAL NOTE: We have assumed Binet Extensions,

\[ F_x = \frac{\alpha^x - \beta^x}{\alpha - \beta}, \quad L_x = \alpha^x + \beta^x, \]

in the calculations of \( x_2, x_3, \cdots, x_5 \) since we are concerned with neighborhoods of zero.

(1) As \( n \to \infty \), \( \frac{F_{n+1}}{\alpha^n} \to (\alpha - \beta)^{-1} \). Let \( p_n = F_{n+1} \) and \( q_n = F_n \).

Then, as \( n \to \infty \),

\[ \frac{F_{n+1}}{\alpha^n} \to (\alpha - \beta)^{-1} \quad \text{and} \quad -\frac{q_n}{\alpha^n} \to (\alpha - \beta)^{-\alpha}. \]

Since \( \alpha q_n - p_n \to 0 \), we have \( x_1 = (\alpha - \beta)^{\alpha - 1} \approx 5(\alpha - 1)/2 \approx 5 \cdot 309 \approx 1.644 \).

For \( x \) real, we define \( L_x = \alpha^x + \beta^x \) and \( F_x = (\alpha^x - \beta^x)/(\alpha - \beta) \).

Let \( Y_i, i = 2, 3, 4, 5 \), denote the limits without absolute value signs; then \( X_i = |Y_i| \).

(2) Using L'Hospital's rule, we have (since \( \alpha \beta = -1 \)),

\[ X_4 = \lim_{n \to 0} \left| \frac{F_{n+1}}{n^{m-1}F_n} \right| \]

\[ X_5 = \lim_{n \to 0} \left| \frac{L_n}{n} - 2 \right| \]
\[ Y_2 = \lim_{x \to 0} \left( \frac{F}{x^m} \right) = \frac{\log \alpha - \log \beta}{\alpha - \beta} = \frac{2 \log \alpha - i\pi}{\alpha - \beta}, \]

where \( i^2 = -1 \), and \( \log (-1) = i\pi \), using principal values. Thus,

\[ X_2 = |Y_2| = \sqrt{\left(4 \log^2 \alpha + \pi^2\right)/5}. \]

(3) Using L'Hospital's rule, we have

\[ Z_3 = \lim_{x \to 0} \frac{x}{F_x} = \frac{\alpha - \beta}{\log \alpha - \log \beta} = \frac{\alpha - \beta}{2 \log \alpha - i\pi}. \]

Thus,

\[ Y_3 = \lim_{x \to 0} \left( \frac{F}{x^{m-1} F_x} \right) = Y_2 \cdot Z_3^m, \]

and so

\[ X_3 = |Y_3| = |Y_2| \cdot |Z_3|^m = X_2 |Z_3|^m = \left(\frac{4 \log^2 \alpha + \pi^2}{5}\right)^{(m-1)/2}. \]

(4) We readily find that

\[ Y_4 = \lim_{x \to 0} \left( \frac{F}{x^{m-2}} \right) = Y_2 \cdot Z_3 = 1, \]

and so \( X_4 = 1 \).

(5) Using L'Hospital's rule, we have

\[ Y_5 = \lim_{x \to 0} \frac{(L - 2)/x}{x} = \log \alpha + \log \beta = i\pi, \]

and so \( X_5 = |Y_5| = \pi \).

Also partially solved by the Proposer, and also solved by M. Yoder and D. Jaiswal.
Shades of Euler

H-149 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee.

For $s = \sigma + it$, let

$$P(s) = \Sigma p^{-s}$$

where the summation is over the primes. Set

$$\sum_{n=1}^{\infty} a(n)n^{-s} = [1 + P(s)]^{-1}$$

$$\sum_{n=1}^{\infty} b(n)n^{-s} = [1 - P(s)]^{-1}$$

Determine the coefficients $a(n)$ and $b(n)$.

Solution by the Proposer.

For $n = p_1^{a_1} \cdots p_m^{a_m}$ let $\rho(n) = a_1 + \cdots + a_m$ and $\lambda(n) = (-1)^{\rho(n)}$.

We claim that

$$a(n) = a \left( \frac{a_1 \cdots a_m}{p_1^{a_1} \cdots p_m^{a_m}} \right) \frac{\lambda(n)(a_1 + \cdots + a_m)!}{a_1! \cdots a_m!}$$

and that $b(n) = |a(n)|$.

The proof is by induction on $\rho(n)$. If $\rho(n) = 1$, $n$ is prime and we have $a(n) + a(1) = 0$ and the validity of the assertion is obvious. Since in general, we have

$$a(n) + a(n/p_1) + \cdots + a(n/p_m) = 0$$

the result follows by induction. A similar method works for $b(n)$, except that here we have
b(n) - b(n/p_1) - \cdots - b(n/p_m) = 0.

Also solved by L. Carlitz, D. Lind, D. Klanner, and M. Yoder.

TRIPLE THREAT


Show that

$$\sum_{p=1}^{n-1} \sum_{q=1}^{p} \sum_{r=1}^{q} F_{2r-1}^2 = F_{4n} + (n/3)(5n^2 - 14),$$

where $F_n$ is the $n$th Fibonacci number.

Solution by the Proposer.

To establish this result, we need the following identities which have already been established earlier (Fibonacci Quarterly, December, 1966, pp. 369-372):

$$5(F_1^2 + F_2^2 + \cdots + F_{2n-1}^2) = F_{4n} + 2n$$
$$F_4 + F_6 + \cdots + F_{4n} = F_{2n}F_{2n+2}$$
$$5(F_2F_4 + F_4F_6 + \cdots + F_{2n-2}F_{2n} = F_{4n} - 3n$$

Hence,

$$5 \sum_{1}^{q} F_{2r-1}^2 = F_{4q} + 2q.$$

Or,
\[ 5 \sum_{q=1}^{p} \sum_{r=1}^{q} F_{2r-1}^2 = \sum_{1}^{p} F_{4q}^2 + 2 \sum_{1}^{p} q = F_{2p} F_{2p+2} + (p + 1)p. \]

Hence,

\[ 25 \sum_{p=1}^{n-1} \sum_{q=1}^{p} F_{2r-1}^2 = 5 \sum_{1}^{n-1} F_{2p} F_{2p+2} + 5 \sum_{1}^{n-1} p^2 + 5 \sum_{1}^{n-1} p = \]

\[ = F_{4n} - 3n + (5/6)n(n - 1)(2n - 1) + (5/2)n(n - 1) = \]

\[ = F_{4n} + (n/3)(5n^3 - 14). \]

Also solved by C. Peck, M. Yoder, A. Shannon, S. Hamelin, and D. Jaiswal.

EDITORIAL NOTE. C. B. A. Peck, in his solution, obtained the identity

\[ 25 \sum_{q=1}^{n} \sum_{r=1}^{q} F_{2r-1}^3 = I_{4n+2} + 5n(n + 1) - 3. \]

[Continued from page 371.]

and also \( \text{ctn arc cos } \varphi = \sin \text{ arc cos } \varphi = \sqrt{\varphi}. \) The results are summarized below.