ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania, 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-172 Proposed by David Englund, Rockford College, Rockford, Illinois.

Prove or disprove the "identity,"

$$F_{kn} = F_n \sum_{t=1}^{\left[\frac{k+1}{2}\right]} (-1)^{(n+1)(t+1)} \binom{k-t}{t-1} L_n^{k-2t+1} ,$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively, and $[x]$ denotes the greatest integer function.

H-173 Proposed by George Ledin, Jr., Institute of Chemical Biology, University of San Francisco, San Francisco, California.

Solve the Diophantine equation,

$$x^2 + y^2 + 1 = 3xy .$$

H-174 Proposed by Daniel W. Burns, Chicago, Illinois.

Let k be any non-zero integer and $\{S_n\}_{n=1}^{\infty}$ be the sequence defined by $S_n = nk$.

Define the Burn's Function, $B(k)$, as follows: $B(k)$ is the minimal value of n for which each of the ten digits, $0, 1, \cdots, 9$, have occurred in at least one S_m where $1 \le m \le n$. For example, $B(1) = 10$, $B(2) = 45$. Does $B(k)$ exist for all k? If so, find an effective formula or algorithm for calculating it.

SOLUTIONS

OLDIES BUT GOODIES

The following problems are still lacking solutions:

H-22	H-46	H-74	H-86	H-94	H-104	H-108	H-115	H-125
H-23	H-60	H-76	H-87	H-100	H-105	H-110	H-116	H-127
H-40	H-61	H-77	H-90	H-102	H-106	H-113	H-118	H-130
H-43	H-73	H-84	H-91	H-103	H-107	H-114	H-122	

GENERATING FUNCTIONS

H-144 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A. Put

$$[(1 - x)(1 - y)(1 - ax)(1 - by)]^{-1} = \sum_{m,n=0}^{\infty} A_{m,n} x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} A_{n,n} x^n = \frac{1 - abx^2}{(1 - x)(1 - ax)(1 - bx)(1 - abx)} .$$

B. Put

$$(1 - x)^{-1}(1 - y)^{-1}(1 - axy)^{-\lambda} = \sum_{m,n=0}^{\infty} B_{m,n} x^m y^n .$$

Show that

$$\sum_{n=0}^{\infty} B_{n,n} x^n = (1 - x)^{-1}(1 - ax)^{-\lambda} \quad .$$

Solution by the Proposer.

Solution, A. We have

$$A_{m,n} = \sum_{i=0}^{m}\sum_{j=0}^{n} a^i b^j = \frac{(1 - a^{m+1})(1 - b^{n+1})}{(1 - a)(1 - b)} \quad ,$$

so that

$$\begin{aligned}
\sum_{n=0}^{\infty} A_{n,n} x^n &= \sum_{n=0}^{\infty} \frac{(1 - a^{n+1})(1 - b^{n+1})}{(1 - a)(1 - b)} x^n \\
&= \frac{1}{(1 - a)(1 - b)} \left\{ \frac{1}{1 - x} - \frac{a}{1 - ax} - \frac{b}{1 - bx} + \frac{ab}{1 - abx} \right\} \\
&= \frac{1}{1 - b} \left\{ \frac{1}{(1 - x)(1 - ax)} - \frac{b}{(1 - bx)(1 - abx)} \right\} \\
&= \frac{1 - abx^2}{(1 - x)(1 - ax)(1 - bx)(1 - abx)} \quad .
\end{aligned}$$

Solution, B. We have

$$(1 - x)^{-1}(1 - y)^{-1}(1 - axy)^{-\lambda}$$

$$= \sum_{r,s,t=0}^{\infty} \frac{(\lambda)_t}{t!} a^t x^{r+t} y^{s+t} \quad ,$$

where

$$(\lambda)_t = (\lambda - 1)(\lambda - 2) \cdots (\lambda - t + 1) \quad (t \geq 1) \quad \text{and} \quad (\lambda)_0 = 1,$$

so that

$$B_{m,n} = \sum_{t=0}^{\min(m,n)} \frac{(\lambda)_t}{t!} a^t .$$

Hence

$$\sum_{n=0}^{\infty} B_{n,n} x^n = \sum_{n=0}^{\infty} x^n \sum_{t=0}^{\infty} \frac{(\lambda)_t}{t!} a^t$$

$$= \sum_{t=0}^{\infty} \frac{(\lambda)_t}{t!} (ax)^t \sum_{n=0}^{\infty} x^n$$

$$= (1 - x)^{-1}(1 - ax)^{-\lambda} .$$

Also solved by M. Yoder and D. Jaiswal.

FACTOR ANALYSIS

H-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

If

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

is the canonical factorization of n, let $\lambda(n) = e_1 + \cdots + e_r$. Show that $\lambda(n) \le \lambda(F_n) + 1$ for all n, where F_n is the n^{th} Fibonacci number.

Solution by the Proposer.

Clearly, $\lambda(mn) = \lambda(m) + \lambda(n)$, and if $m|n$ then $\lambda(m) \le \lambda(n)$. Also, $1 = \lambda(p) \le \lambda(F_p)$ for any prime p. We show by induction that $\lambda(p^k) \le \lambda(F_{p^k})$ for all k, except when $p = k = 2$, when $\lambda(4) = \lambda(F_4) + 1$. The cases when $p^k \le 12$ are checked directly. Assume the result is true for p^{k-1}. Then since $p^k > 12$, by Carmichael's theorem ("On the Numerical

Factors of the Arithmetical Forms $\alpha^n \pm \beta^n$," Annals of Math. (2^{nd} Ser.), 15, pp. 30-70, Theorem XXIII) there is a prime dividing F_{p^k} not dividing $F_{p^{k-1}}$. Then since $F_{p^{k-1}} \mid F_{p^k}$, we have

$$\lambda(F_{p^k}) \ge 1 + \lambda(F_{p^{k-1}}) \ge k ,$$

completing the induction. Hence $\lambda(p^k) \le \lambda(F_{p^k})$ except when $p = k = 2$. In the factorization

$$n = p_1^{e_1} \cdots p_r^{e_r} ,$$

we can assume $p_1 = 2$, and $e_1 = 0$ if necessary. Then

$$F_{p_i^{e_i}}, \cdots, F_{p_r^{e_r}}$$

are pairwise relatively prime since $p_1^{e_1}, \cdots, p_r^{e_r}$ are, and since $F_{p_i^{e_i}}$ divides F_n for each i, so their product

$$F_{p_1^{e_1}} \cdots F_{p_r^{e_r}} \,\Big|\, F_n .$$

Hence,

$$\lambda(F_n) \ge \lambda(F_{p_1^{e_1}} \cdots F_{p_r^{e_r}}) = \lambda(F_{p_1^{e_1}}) + \cdots + \lambda(F_{p_r^{e_r}}) \ge$$

$$\ge (e_1 - 1) + e_2 + \cdots + e_r = \lambda(n) - 1 ,$$

which completes the proof.

Also solved by M. Yoder.

CONVERGING FRACTIONS

H-147 Proposed by George Ledin, Jr., University of San Francisco, San Francisco, California.

Find the following limits. F_k is the k^{th} Fibonacci number, L_k is the k^{th} Lucas number, $\pi = 3.14159\cdots$, $\alpha = (1 + \sqrt{5})/2 = 1.61803\cdots$, $m = 1, 2, 3, \cdots$.

$$X_1 = \lim_{n\to\infty} \frac{F_{F_{n+1}}}{F^{\alpha}_{F_n}}$$

$$X_2 = \lim_{n\to 0} \left|\frac{F_{n^m}}{n^m}\right|$$

$$X_3 = \lim_{n\to 0} \left|\frac{F_{n^m}}{F^m_n}\right|$$

$$X_4 = \lim_{n\to 0} \left|\frac{F_{n^m}}{n^{m-1}F_n}\right|$$

$$X_5 = \lim_{n\to 0} \left|\frac{L_n - 2}{n}\right|$$

Solution by David Zeitlin, Minneapolis, Minnesota.

EDITORIAL NOTE: We have assumed Binét Extensions,

$$F_x = \frac{\alpha^x - \beta^x}{\alpha - \beta}, \quad L_x = \alpha^x + \beta^x ,$$

in the calculations of $x_2, x_3, \cdots, x_5$ since we are concerned with neighborhoods of zero!

(1) As $n\to\infty$, $F_n/\alpha^n \to (\alpha-\beta)^{-1}$. Let $p_n = F_{n+1}$ and $q_n = F_n$. Then, as $n\to\infty$,

$$F_{p_n}\Big/\alpha^{p_n} \to (\alpha - \beta)^{-1} \quad\text{and}\quad -F^{d}_{q_n}\Big/\alpha^{2q_n} \to (\alpha - \beta)^{-\alpha} .$$

Since $\alpha q_n - p_n \to 0$, we have $x_1 = (\alpha - \beta)^{\alpha-1} \cong 5^{(\alpha-1)/2} \cong 5^{.309} \cong 1.644$.

For x real, we define $L_x = \alpha^x + \beta^x$ and $F_x = (\alpha^x - \beta^x)/(\alpha - \beta)$. Let Y_i, $i = 2, 3, 4, 5$, denote the limits without absolute value signs; then $X_i = |Y_i|$.

(2) Using L'Hospital's rule, we have (since $\alpha\beta = -1$) ,

$$Y_2 = \lim_{x\to 0}\left(F_{x^m}\Big/x^m\right) = \frac{\log\alpha - \log\beta}{\alpha - \beta} = \frac{2\log\alpha - i\pi}{\alpha - \beta},$$

where $i^2 = -1$, and $\log(-1) = i\pi$, using principal values. Thus,

$$X_2 = |Y_2| = \sqrt{(4\log^2\alpha + \pi^2)/5} .$$

(3) Using L'Hospital's rule, we have

$$Z_3 = \lim_{x\to 0}\frac{x}{F_x} = \frac{\alpha - \beta}{\log\alpha - \log\beta} = \frac{\alpha - \beta}{2\log\alpha - i\pi} .$$

Thus,

$$Y_3 = \lim_{x\to 0}\left(F_{x^m}\Big/F_x^m\right) = Y_2 \cdot Z_3^m ,$$

and so

$$X_3 = |Y_3| = |Y_2|\cdot|Z_3|^m = X_2|Z_3|^m = \left(\frac{4\log^2\alpha + \pi^2}{5}\right)^{-(m-1)/2} .$$

(4) We readily find that

$$Y_4 = \lim_{x\to 0}\left(F_{x^m}\Big/x^{m-1}F_x\right) = Y_2 \cdot Z_3 = 1 ,$$

and so $X_4 = 1$.

(5) Using L'Hospital's rule, we have

$$Y_5 = \lim_{x\to 0}(L_x - 2)/x = \log\alpha + \log\beta = i\pi ,$$

and so $X_5 = |Y_5| = \pi$.

Also partially solved by the Proposer, and also solved by M. Yoder and D. Jaiswal.

SHADES OF EULER

H-149 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee.

For $s = \sigma + it$, let

$$P(s) = \Sigma p^{-s},$$

where the summation is over the primes. Set

$$\sum_{n=1}^{\infty} a(n)n^{-s} = [1 + P(s)]^{-1},$$

$$\sum_{n=1}^{\infty} b(n)n^{-s} = [1 - P(s)]^{-1}.$$

Determine the coefficients $a(n)$ and $b(n)$.

Solution by the Proposer.

For $n = p_1^{a_1} \cdots p_m^{a_m}$ let $\rho(n) = a_1 + \cdots + a_m$ and $\lambda(n) = (-1)^{\rho(n)}$. We claim that

$$a(n) = a\left(p_1^{a_1} \cdots p_m^{a_m}\right) = \frac{\lambda(n)(a_1 + \cdots + a_m)!}{a_1! \cdots a_m!}$$

and that $b(n) = |a(n)|$.

The proof is by induction on $\rho(n)$. If $\rho(n) = 1$, n is prime and we have $a(n) + a(1) = 0$ and the validity of the assertion is obvious. Since in general, we have

$$a(n) + a(n/p_1) + \cdots + a(n/p_m) = 0,$$

the result follows by induction. A similar method works for $b(n)$, except that here we have

$$b(n) - b(n/p_1) - \cdots - b(n/p_m) = 0 .$$

Also solved by L. Carlitz, D. Lind, D. Klarner, and M. Yoder.

TRIPLE THREAT

H-150 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Show that

$$25 \sum_{p=1}^{n-1} \sum_{q=1}^{p} \sum_{r=1}^{q} F_{2r-1}^2 = F_{4n} + (n/3)(5n^2 - 14) ,$$

where F_n is the n^{th} Fibonacci number.

Solution by the Proposer.

To establish this result, we need the following identities which have already been established earlier (Fibonacci Quarterly, December, 1966, pp. 369-372):

$$5(F_1^2 + F_3^2 + \cdots + F_{2n-1}^2) = F_{4n} + 2n$$

$$F_4 + F_8 + \cdots + F_{4n} = F_{2n}F_{2n+2}$$

$$5(F_2F_4 + F_4F_6 + \cdots + F_{2n-2}F_{2n} = F_{4n} - 3n$$

Hence,

$$5 \sum_1^q F_{2r-1}^2 = F_{4q} + 2q .$$

Or,

$$5\sum_{q=1}^{p}\sum_{r=1}^{q} F_{2r-1}^2 = \sum_{1}^{p} F_{4q} + 2\sum_{1}^{p} q = F_{2p}F_{2p+2} + (p+1)p\ .$$

Hence,

$$25\sum_{p=1}^{n-1}\sum_{q=1}^{p}\sum_{r=1}^{q} F_{2r-1}^2 = 5\sum_{1}^{n-1} F_{2p}F_{2p+2} + 5\sum_{1}^{n-1} p^2 + 5\sum_{1}^{n-1} p =$$

$$= F_{4n} - 3n + (5/6)n(n-1)(2n-1) + (5/2)n(n-1) =$$

$$= F_{4n} + (n/3)(5n^2 - 14)\ .$$

Also solved by C. Peck, M. Yoder, A. Shannon, S. Hamelin, and D. Jaiswal.

EDITORIAL NOTE. C. B. A. Peck, in his solution, obtained the identity

$$25\sum_{q=1}^{n}\sum_{r=1}^{q} F_{2r-1}^2 = L_{4n+2} + 5n(n+1) - 3.$$

[Continued from page 371.]

and also ctn arc cos φ = sin arc cos $\varphi = \sqrt{\varphi}$. The results are summarized below.