

AN EXTENSION OF A THEOREM OF EULER

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Leonhard Euler, the great mathematician of the 18th Century, wrote some of the greatest works ever read by man. Among the numerous mathematical interests of this genius was the study of the problem of partitions.

A partition of an arbitrary positive integer, say n , is a representation of n as the sum of any number of integral parts. For example, the number 6 has 11 partitions, since

$$6 = 5 + 1 = 4 + 2 = 4 + 1 + 1 = 3 + 3 = 3 + 2 + 1 = 3 + 1 + 1 + 1 \\ = 2 + 2 + 2 = 2 + 2 + 1 + 1 = 2 + 1 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1 + 1 .$$

(Problem. Show that the number of partitions of 7 is equal to 15.)

It is apparent that to find the partition of some positive integer n using the same methods we used to find the solution for $n = 6$ above is a clumsy and difficult procedure. To overcome this difficulty, Euler combined the partition with a generating function which led to his discovering a powerful recurrence formula with which to attack the study of partitions.

(For those readers who wish to discover how Euler found his recurrence formula, the author recommends they read a chapter on partitions in any good book on number theory. One such book is An Introduction to the Theory of Numbers by Niven and Zuckerman, published by John Wiley and Sons, Inc.)

EULER'S FAMOUS RECURRENCE FORMULA is usually written as follows:

$$(1) \quad p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) + \dots \\ + (-1)^{k+1} p(n - \frac{1}{2}k(3k - 1)) + (-1)^{k+1} p(n - \frac{1}{2}k(3k + 1)) + \dots ,$$

where $p(0) = 1$ and $n, k = 1, 2, 3, \dots$.

To understand the above theorem (1), let us go back to the beginning of this paper where we have shown that the number $n = 6$ has 11 partitions.

Then, what the formula says in a very precise way is that $p(6) = 11$, or more generally, $p(n)$ is the number of ways an arbitrary positive integer, say n , can be partitioned into equal or distinct parts.

Now that we know what $p(n)$ means, let us find out how to use the formula (1) and in so doing, it will become evident how greatly the genius of Euler reduced the work required in order to find a solution for each $p(n)$ ($n = 0, 1, 2, \dots$).

We define $p(0) = 1$ so that in (1) when $n = 1$, we have

$$p(1) = p(1 - 1) = p(0) = 1 ,$$

and now that we have found a value for $p(1)$, we are in a position to find a value (or the number of partitions) for $p(2)$ since for $n = 2$, we have

$$p(2) = p(2 - 1) + p(2 - 2) = p(1) + p(0) = 1 + 1 = 2 .$$

We continue in this exact way, using the information that $p(0) = 1$, $p(1) = 1$, and $p(2) = 2$ to find a value for $p(3)$ and then step-by-step values for $p(4)$, $p(5)$, $p(6)$, and so on.

In the following are examples of how to find values for the $p(n)$ when $n = 1, 2, 3, \dots, 7$ by using Leonhard Euler's very important recurrence formula (in (1)): We define $p(0) = 1$, then

$$p(1) = p(1 - 1) = p(0) = 1,$$

$$p(2) = p(2 - 1) + p(2 - 2) = p(1) + p(0) = 1 + 1 = 2,$$

$$p(3) = p(3 - 1) + p(3 - 2) = p(2) + p(1) = 2 + 1 = 3,$$

$$p(4) = p(4 - 1) + p(4 - 2) = p(3) + p(2) = 3 + 2 = 5,$$

$$p(5) = p(5 - 1) + p(5 - 2) - p(5 - 5) = p(4) + p(3) - p(0) = 5 + 3 - 1 = 7,$$

$$p(6) = p(6 - 1) + p(6 - 2) - p(6 - 5) = p(5) + p(4) - p(1) = 7 + 5 - 1 = 11, .$$

(Compare this way of finding $p(6) = 11$ with the way we showed that the number $n = 6$ has 11 partitions at the beginning of the paper, and you will

realize the magnificence of Euler's formula as a systematized labor-saving device, especially for large n where his formula is really needed.)

For our last example, we find $p(7)$:

$$\begin{aligned} p(7) &= p(7 - 1) + p(7 - 2) - p(7 - 5) - p(7 - 7) = \\ &= p(6) + p(5) - p(2) - p(0) = 11 + 7 - 2 - 1 = 15. \end{aligned}$$

Continuing in this exact way, we may, of course, step-by-step find values for $p(8)$, $p(9)$, $p(10)$, \dots , $p(n)$, \dots , where n runs through the positive integers $n = 8, 9, 10, \dots$, to infinity.

It is evident that Euler's great recurrence formula (1) systematized the study of partitions. However, to determine the values of $p(n)$ for still large n required an enormous amount of work. (For example, to show that $p(243) = 133978259344888$, we must first find a value for each $p(n)$ ($n = 0, 1, 2, \dots, 242$) from $p(0)$ through $p(242)$ inclusive.) To this end, in what follows of this paper, we show how to greatly reduce the work required in finding values of the $p(n)$ by applying a new theorem from a paper entitled "Recurrence Formulas," by Joseph Arkin and Richard Pollack (The Fibonacci Quarterly, Vol. 8, No. 1, February, 1970, pp. 4-5).

In fact, using formula (1) of "Recurrence Formulas" and applying the method that has been found by this author to formula (1) so greatly reduces the work involved that to find the value of, say, $p(243)$, it would only be necessary to know the value of each $p(0)$ through $p(122)$. The reduction in work is evident, since in Euler's recurrence formula, to find a value for $p(243)$, we must first find a value for each $p(0)$ through $p(242)$.

To explain the new method of determining the value of any partition $(p(n))$, we shall, as examples, find the values of $p(16)$ and $p(17)$.

Then, to find the value of $p(16)$, we set up the table on the next page.

EXPLANATION OF HOW TABLE WAS MADE

1. On row A we have placed the values of $(-1)^{k+1} \frac{1}{2}k(3k \mp 1)$ for $k = 1, 2, 3$.
2. We then take half of 16 to get 8 and so we write under column a the consecutive numbers from 8 through 16.
3. Now, next to the number 8 (under column a) we place under column b the value $p(16 - 8)$, next to the number 9 (under a) we place under column b the value $p(16 - 9)$ and so on to complete column b with $p(16 - 16) = p(0)$.

	A	1	2	-5	-7	12	15
<u>a</u>	<u>b</u>						
8	p(8)	p(7)	p(6)	-p(3)	-p(1)		
9	p(7)		p(7)	-p(4)	-p(2)		
10	p(6)			-p(5)	-p(3)		
11	p(5)			-p(6)	-p(4)		
12	p(4)			-p(7)	-p(5)	p(0)	
13	p(3)				-p(6)	p(1)	
14	p(2)				-p(7)	p(2)	
15	p(1)					p(3)	p(0)
16	p(0)					p(4)	p(1)

4. We fill in the rest of the table in the same way we plot a graph. For example, the $p(7)$ under the column where $A = 1$ and on the row where $\underline{a} = 8$ is said to be in box $(8,1)$ or more generally this $p(7)$ is found in box (\underline{a}, A) where $\underline{a} = 8$ and $A = 1$. Now, it will be observed that in each box (\underline{a}, A) we find the term $p(\underline{a} - A)$, or in box $(\underline{a}, -A)$ we find the term $-p(\underline{a} - A)$ except that there are no terms $\mp p(\underline{a} - A)$ entered at all when $\frac{1}{2} \cdot 16 = 8 \leq \underline{a} - A < 0$.

Let us consider a few numerical examples of what was just said. Written into the five boxes $(8,1)$, $(8,2)$, $(13,-7)$, $(13,15)$, and $(16,7)$ we find respectively the following: $p(8 - 1) = p(7)$, $p(8 - 2) = p(6)$, $-p(13 - 7) = -p(6)$, no entry (since $13 - 15 = -2 < 0$) and no entry (since $16 - 7 = 9 > 8$).

Now that we have filled in the table, we then multiply each partition under the column \underline{b} in the table together with the sum of the partitions directly to its right and on the same row. We then have the following products (row-by-row):

$$\begin{aligned}
 & p(8)[p(7) + p(6) - p(3) - p(1)] \\
 & p(7)[p(7) - p(4) - p(2)] \\
 & p(6)[-p(5) - p(3)] \\
 & p(5)[-p(6) - p(4)] \\
 & p(4)[-p(7) - p(5) + p(0)] \\
 & p(3)[-p(6) + p(1)] \\
 & p(2)[-p(7) + p(2)]
 \end{aligned}$$

$$\begin{aligned} & p(1)[p(2) + p(0)] \\ & p(0)[p(4) + p(1)] \end{aligned}$$

and after replacing the $p(\)$ with their numerical values, we find $p(16)$ as follows:

$$\begin{aligned} 22(15 + 11 - 3 - 1) &= 22 \cdot 22 = 484 \\ 15(15 - 5 - 2) &= 15 \cdot 8 = 120 \\ 11(-7 - 3) &= -11 \cdot 10 = -110 \\ 7(-11 - 5) &= -7 \cdot 16 = -112 \\ 5(-15 - 7 + 1) &= -5 \cdot 21 = -105 \\ 3(-11 + 1) &= -3 \cdot 10 = -30 \\ 2(-15 + 2) &= -2 \cdot 13 = -26 \\ 1(3 + 1) &= 1 \cdot 4 = 4 \\ 1(5 + 1) &= 1 \cdot 6 = \underline{6} \\ \text{Total } 231 &= p(16) . \end{aligned}$$

To find a numerical value for $p(17)$, we use the exact methods that were used to find a numerical value for $p(16)$. The important difference is that since 17 is an odd number, we must then take half of $17 - 1$ and then complete the following table using the same methods that we used to complete a table for $p(16)$ (that is, we shall begin by writing under column a the consecutive numbers from 8 through 17, etc.).

To find the value of $p(17)$, we erect the following table:

A	1	2	-5	-7	12	15
<u>a</u>						
8	<u>b</u>					
9	$p(9)$	$p(7)$	$p(6)$	$-p(3)$	$-p(1)$	
10	$p(8)$		$p(7)$	$-p(4)$	$-p(2)$	
11	$p(7)$			$-p(5)$	$-p(3)$	
12	$p(6)$			$-p(6)$	$-p(4)$	
13	$p(5)$			$-p(7)$	$-p(5)$	$p(0)$
14	$p(4)$				$-p(6)$	$p(1)$
15	$p(3)$				$-p(7)$	$p(2)$
16	$p(2)$					$p(3)$
17	$p(1)$					$p(0)$
	$p(0)$					$p(4)$
						$p(1)$
						$p(5)$
						$p(2)$

We now multiply each partition under column \underline{b} in the table together with the sum of the partitions directly to its right and on the same row to get the following products:

$$\begin{aligned} & p(9)[p(7) + p(6) - p(3) - p(1)] \\ & p(8)[p(7) - p(4) - p(2)] \\ & p(7)[-p(5) - p(3)] \\ & p(6)[-p(6) - p(4)] \\ & p(5)[-p(7) - p(5) + p(0)] \\ & p(4)[-p(6) + p(1)] \\ & p(3)[-p(7) + p(2)] \\ & p(2)[p(3) + p(0)] \\ & p(1)[p(4) + p(1)] \\ & p(0)[p(5) + p(2)] \end{aligned}$$

and after replacing the $p(\)$ with their numerical values, we find $p(17)$ as follows:

$$\begin{aligned} 30(15 + 11 - 3 - 1) &= 30 \cdot 22 = 660 \\ 22(15 - 5 - 2) &= 22 \cdot 8 = 176 \\ 15(-7 - 3) &= -15 \cdot 10 = -150 \\ 11(-11 - 5) &= -11 \cdot 16 = -176 \\ 7(-15 - 7 + 1) &= -7 \cdot 21 = -147 \\ 5(-11 + 1) &= -5 \cdot 10 = -50 \\ 3(-15 + 2) &= -3 \cdot 13 = -39 \\ 2(3 + 1) &= 2 \cdot 4 = 8 \\ 1(5 + 1) &= 1 \cdot 6 = 6 \\ 1(7 + 2) &= 1 \cdot 9 = \underline{9} \\ \text{Total} & 297 = p(17) . \end{aligned}$$

In conclusion, it may be interesting to mention that we could have used smaller $p(\)$ to find $p(16)$ and $p(17)$. For example: Since $p(0)$ through $p(8)$ will determine $p(16)$ and by the methods used in this paper, it is evident that the numerical values of $p(0)$ through $p(4)$ will enable us to find values for $p(0)$ through $p(8)$ then we need only have used the values of the

partitions $p(0)$ through $p(4)$ to find $p(16)$. This reduction rule is applicable in finding the value of any $p(n)$, however it defeats the purpose of the method to reduce too much.

Of course, applying the method of this paper to find small partitions like $p(16)$ or $p(17)$ does not show the method to its fullest — but when used to find a value for large partitions, like say, $p(243) = 133978259344888$, the method shown in this paper very greatly reduces the work involved.



[Continued from p. 364.]

and the induction is complete. Thus the C-array is precisely the B-array. Thus, $B_{m,p} \equiv C_{m,p}$, and further, the pattern observed by Umansky and Karst persists for all $n \geq 1$, $m \geq 2$. The case $m = 1$ was earlier verified.

Theorem 2 (Independent). If one ignores the signs of the coefficients in Array C, then the sum across the m^{th} row is L_m .

Proof. Interchange the first column on the right with the column on the left and set $n = 1$. The left column is now $-L_m$ and all of the terms on the right are negative. Equality still holds since Theorem 1 is true. Thus

$$1 + \sum_{j=1}^{[m/2]} C_{m,j} = \sum_{j=0}^{[m/2]} C_{m,j} = L_m .$$

REFERENCES

1. Mark Feinberg, "Lucas Triangle," Fibonacci Quarterly, Vol. 5, No. 5, Dec., 1967, pp. 486-490.
2. Harlan Umansky, "Letters to the Editor," Fibonacci Quarterly, Vol. 8, No. 1, Feb., 1970, p. 89.
3. V. E. Hoggatt, Jr., "Convolution Triangles for Generalized Fibonacci Numbers," Fibonacci Quarterly, Vol. 8, No. 2, Mar., 1970, pp. 158-171.

