THE "DIFFERENCE SERIES" OF MADACHY

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In a recent issue, Madachy [1] has raised two conjectures concerning difference series which result at the various levels in the Sieve of Eratosthenes; both conjectures are valid. In this modified sieve, the sieving prime is discarded along with its multiples. The difference series associated with a prime is the series of differences which occur between the members of the remaining list after sievings through that particular prime.

<u>Conjecture 1.</u> If d $(d > 2)$ is the number of terms in one period of the difference series, then the series is symmetrical about the $(d/2)$th term.

<u>Conjecture 2.</u> The $(d/2)^{th}$ term $(d > 2)$ will always be 4.

The validity of Conjecture 1 can be argued as follows. Consider the numbers which remain after we have sieved by $2, 3, 5, \cdots, p_n$ and let $P = 2 \cdot 3 \cdot 5 \cdots p_n$. Then, as Madachy shows, there are $P + 1$ numbers in the original set of consecutive integers from which one period of the difference series is formed; we note that both endpoints must be included in order to generate the differences. Starting from $P - 1$ of these integers, sieving by 2 can be done by sieving each second number and beginning at either end, since 2 divides P. The numbers 0 and P play like roles, as do the numbers 1 and $P - 1$. Similarly, since 3 divides P we can again count from either end of the original set and sieve each third number. Likewise we do this for each number through p_n. To illustrate, consider $n = 3$ so that $P - 1 = 29$. We underline to indicate sieving.

$$29 \;\; \underline{28} \;\; \underline{27} \;\; \underline{26} \;\; \underline{25} \;\; \underline{\underline{24}} \;\; 23 \;\; \underline{22} \;\; \underline{21} \;\; \underline{\underline{20}} \;\; 19 \;\; \underline{\underline{18}} \;\; 17 \;\; \underline{16} \;\; \underline{\underline{15}} \;\; \underline{14} \;\; 13 \;\; \cdots$$

Thus the difference series begins $6, 4, 2, 4, \cdots$. The last element of a period of the difference series occurs from the pair $P - 1$, $P + 1$, and is always 2. Since we can apply the process forward or backward over the first period (deleting this last difference), the differences are symmetrically arranged about the "middle difference," i.e., about the $(d/2)^{th}$ term.

Conjecture 2 can be settled by considering how this "middle difference" must be formed. Since $d > 2$, the middle difference must span the number $P/2$ in the original set of consecutive integers, which is an odd number; actually $P/2 = 3 \cdot 5 \cdots p_n$. Since $(P/2) + 1$ and $(P/2) - 1$ are even, this "middle difference" must be at least 4. From the results of Conjecture 1, symmetry then shows that if it is not 4, it must be at least 8. In this case, both of the odd numbers $(P/2) + 2$ and $(P/2) - 2$ must have been sieved out at some stage; i.e., by some number $3, 5, \cdots, p_n$. However, if we consider a decomposition of $(P/2) \pm 2$, we see that this is not possible, for

$$(P/2) \pm 2 = 3 \cdot 5 \cdots p_n \pm 2 ,$$

and the remainder is ± 2 when it is divided by any of these primes, hence it could not have been sieved out.

It is of interest to note that no sieving with any prime p greater than 3, although the actual numbers which are sieved out are not regularly arranged, a ratio of exactly $1/p$ of them disappear. In terms of the difference series this means that within one period of that series formed after sieving by $2, 3, 5, \cdots, p$, exactly $1/p$ of the pairs of members of the previous difference series are combined. An example for $P = 30$ illustrates this.

$$\begin{array}{l} \underline{4\ \ 2}\ \ 4\ \ 2\ \ 4\ \ 2\ \ 4\ \ \underline{2\ \ 4}\ \ 2 \mid \underline{4\ \ 2}\ \ 4\ \ 2\ \cdots \\ \ \ \ 6\ \ \ \ 4\ \ 2\ \ 4\ \ 2\ \ 4\ \ \ \ 6\ \ \ \ 2 \mid \ 6\ \ \ \ 4\ \ 2\ \cdots \end{array}$$

From the 10 differences (of 5 periods) of the previous series we form 2 new differences, a ratio of $2/10 = 1/5$ for the sieving prime 5.

A very difficult problem is to try to determine in general exactly which pairs of differences are to be combined to form the next difference series. In the above example, pairs numbered $1, 8, 11, 18, \cdots$ are combined. For the next sequences we combine pairs numbered:

(7) 1, 13, 20, 24, 31, 35, 42, 54; 1 + 56, ⋯
(11) 1, 27, 32, 42, 47, 58, 73, 77, 93, 103, ⋯
(13) 1, 35, 44, 51, 62, 77, 84, 99, 110, 115, ⋯
(17) 1, 56, 62, 75, 94, 100, 119, 132, 139, ⋯ .

No general formula for the n^{th} difference series seems to exist.

REFERENCE

1. Joseph S. Madachy, "Recreational Mathematics — 'Difference Series' Resulting from Sieving Primes," Fibonacci Quarterly, 7 (1969), pp. 315-318.

[Continued from p. 346.]

If n is not divisible by 11, 13, or 17, then $p_2 < 19 \leqq p_3$. Taking $q = 19$ in (3.0), we have

$$\sum_{p|n} \frac{1}{p} > \frac{1}{3} + \frac{1}{5} + \frac{\log (16/15)}{19 \log (19/18)} > \frac{1}{3} + \frac{1}{5} + \frac{1}{17} + \frac{\log (256/255)}{257 \log (257/256)} .$$

This completes the proof of the lower bound for (C) and also that of the new parts of Theorem 1.

REFERENCES

1. L. E. Dickson, History of the Theory of Numbers, Vol. 1, New York, 1934.
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3. M. Perisastry, "A Note on Odd Perfect Numbers," Math. Student, 26 (1958), 179-181.
4. D. Suryanarayana and N. Venkateswara Rao, "On Odd Perfect Numbers," Math. Student, 29 (1961), 133-137.
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6. J. Touchard, "On Prime Numbers and Perfect Numbers," Scripta Math., 19 (1953), 35-39.