COMPLETE DIOPHANTINE SOLUTION OF THE PYTHAGOREAN TRIPLE (a, b = a + 1,c)

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In connection with problem B-123 (Fibonacci Quarterly 5 (1967), p. 288) the question was raised whether Pell numbers provide the only possible Diophantine solutions for the Pythagorean triple (a, b = a + 1, c). To prove that this is indeed so, it is necessary and sufficient to show that the general solution for the Pythagorean triple, when modified for this special case, acquires a form identical with relationships that are characteristic for Pell numbers, P_n.

The proof is based on a property of a class of sequences, of which both the Fibonacci and the Pell sequences are particular cases. By applying the recursion formula that is specific to the Pell sequence, an identity for the general sequence is transformed into one that is valid only for the Pell sequence. It is precisely this identity that must be satisfied by the special Pythagorean triples.

We start with the single-membered, purely periodic, infinite continued fraction

$$\frac{1}{g+}\cdots = -g + \frac{\sqrt{g^2+4}}{2} ,$$

where g is a positive integer. The limiting value shown is the positive root, x_1, of $x^2 + gx - 1 = 0$. The numerators and denominators of the convergents are obtained by the well-known recursion formula

(1) $$G_{n+2} = gG_{n+1} + G_n \qquad (G_0 = 0,\ G_1 = 1) .$$

These numbers appear in the powers of, e.g.,

$$-x_2 = \frac{g}{2} + \frac{\sqrt{g^2+4}}{2} ,$$

$$(2)\qquad (-x_2)^{n+3} = G_{n+2} + G_{n+3}x_2 = \left(\frac{g}{2}G_{n+3} + G_{n+2}\right) + G_{n+3}\frac{\sqrt{g^2+4}}{2} .$$

Now

$$(3)\qquad \left(\frac{g}{2}G_{n+3} + G_{n+2}\right)^2 = \frac{g^2}{4}G_{n+3}^2 + gG_{n+3}G_{n+2} + G_{n+2}^2 ,$$

and

$$(4)\qquad \left(G_{n+3}\frac{\sqrt{g^2+4}}{2}\right)^2 = \frac{g^2}{4}G_{n+3}^2 + gG_{n+3}G_{n+2} + G_{n+3}G_{n+1} .$$

By subtracting (4) from (3) and using (1), we obtain

$$(5)\qquad -(G_{n+1}^2 - G_{n+2}G_n) = G_{n+2}^2 - G_{n+3}G_{n+1}$$

(the left side being simply a transformation of the right).

Since (5) holds for all values of n, each side must equal the same constant. Upon substituting $G_0 = 0$ and $G_1 = 1$ on the left (note that the value of $G_2 = g$ is not really needed), we find $-(1^2 - g\cdot 0) = -1$, and hence

$$(6)\qquad G_{n+2}G_n - G_{n+1}^2 = (-1)^{n+1} = G_n^2 + g\,G_{n+1}G_n - G_{n+1}^2 .$$

When $g = 1$, the left side reduces to the well-known Fibonacci-number identity

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1} .$$

When $g = 2$, the right side of (6) can be rewritten as the Pell-number identity

$$(P_n + P_{n+1})^2 - 2P_{n+1}^2 = (-1)^{n+1} ,$$

whence

$$P_n + P_{n+1} = \sqrt{2P_{n+1}^2 + (-1)^{n+1}} \tag{7}$$

Also, since the recursion formula for Pell numbers is

$$P_{n+2} = 2P_{n+1} + P_n \quad (P_0 = 0, \quad P_1 = 1)$$

from (1) above, we have

$$P_{n+2} - P_{n+1} = P_{n+1} + P_n \; . \tag{8}$$

These equations have thus been shown to be characteristic for Pell numbers.

The Diophantine solution for the Pythagorean triple, with legs a and b and hypotenuse c, is $2pq = a$ or b, $p^2 - q^2 = b$ or a, and $p^2 + q^2 = c$, where p and q are positive integers of different parity and $p > q$. When $b = a + 1$, $p^2 - q^2 - 2pq = \pm 1$. Solving for p and q, and rearranging terms,

$$p + q = \sqrt{2p^2 \pm 1} \tag{9}$$

$$p - q = \sqrt{2q^2 \pm 1} \; . \tag{10}$$

Since (7) and (8) are obviously equivalent to (9) and (10), p and q must be Pell numbers. In fact, when $q = P_n$ then $p = P_{n+1}$. The even leg of the triangle is

$$2P_{n+1}P_n = a_n \text{ or } b_n \; ,$$

the odd leg,

$$P_{n+1}^2 - P_n^2 = b_n \text{ or } a_n \; ,$$

and the hypotenuse,

$$P_{n+1}^2 + P_n^2 = P_{2n+1} = c_n .$$

Also, the smaller leg is

$$\sum_{m=1}^{2n} P_m = a_n \text{ or } b_n .$$

Except for the lowest nontrivial value 3, the values for both legs are obviously composite numbers.

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TABLE 2

$f_1 = e_1 \qquad f_2 = e_2 \qquad f_3 = e_1e_2 - e_3$

$$f_4 = e_3 - e_1e_2 - e_1e_3 + e_4 + e_2\binom{-e_1}{2} \qquad f_5 = -e_1e_2 + e_3 - e_2e_3 + e_5 + e_1\binom{-e_2}{2}$$

$$f_6 = e_1e_2 - e_3 + 2e_1e_3 - 2e_4 - 2e_2\binom{-e_1}{2} + e_1e_4 - e_6 - e_2\binom{-e_1}{3} - e_3\binom{-e_1}{2}$$

$$f_7 = e_1e_2 - e_3 + e_1e_3 - e_4 - e_2\binom{-e_1}{2} + e_2e_3 - e_5 - e_1\binom{-e_2}{2} + e_1e_5 - e_7 + e_2e_4 - e_1e_2e_3 + \binom{-e_1}{2}\binom{-e_2}{2}$$

$$f_8 = e_1e_2 - e_3 + 2e_2e_3 - 2e_5 - 2e_1\binom{-e_2}{2} + e_2e_5 - e_8 - e_3\binom{-e_2}{2} - e_1\binom{-e_2}{3}$$

TABLE 3

$h_1 = e_2 \qquad h_2 = e_1 \qquad h_3 = e_1e_2 - e_3$

$$h_4 = -e_5 + e_1\binom{e_2}{2} \qquad h_5 = -e_4 + e_2\binom{e_1}{2} \qquad h_6 = -e_8 + e_1\binom{e_2}{3}$$

$$h_7 = -e_7 + \binom{e_1}{2}\binom{e_2}{2} \qquad h_8 = -e_6 + e_2\binom{e_1}{3}$$