ONE-ONE CORRESPONDENCES BETWEEN THE SET N OF POSITIVE INTEGERS AND THE SETS N^n AND $\bigcup_{n\in N} N^n$

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1. Let N be the set of positive integers and let N^n be the set of all n-tuples of positive integers. It is well known that there exist one-one correspondences between N^n and N for all N, and between $\bigcup_{n\in N} N^n$ and N. In this paper, we give examples of such functions.

2. <u>Theorem 1.</u> Define f_n: $N^n \to N$ by

$$(1) \qquad f_n(x_1, x_2, \cdots, x_n) = \binom{s_n}{n} - \sum_{k=1}^{n-1} \binom{s_k - 1}{k} \quad ,$$

where

$$s_k = \sum_{i=1}^{k} x_i$$

for $k \le n$ and the combinatorial symbol $\binom{m}{k}$ is defined to be 0 if $m < k$. Then f_n is a one-one correspondence.

<u>Proof.</u> We begin by defining a relation $\prec$ on N^n as follows:

<u>Definition.</u> $(x_1', x_2', \cdots, x_n') \prec (x_1, x_2, \cdots, x_n)$ if and only if $s_n' < s_n$, or $s_n' = s_n$ and there exists $k \le n$ such that $x_k' < x_k$ and $x_i' = x_i$ for $k < i \le n$.

It is readily established that $\prec$ well-orders N^n. For $\alpha \in N^n$, let $M_\alpha = \{\beta \in N^n \mid \beta \preceq \alpha\}$ and let $f_n(\alpha) = \#(M_\alpha)$ where $\#(M_\alpha)$ is the number of elements in M_α. Since M_α is a finite set, it follows that f_n is a one-one mapping from N^n onto N. We prove by induction on n that $f_n(\alpha)$ is given by (1).

If $n = 1$, we have $f_1(x_1) = \#\{\beta \in N \mid \beta \le x_1\} = x_1$ which is the value (1) gives for $f_1(x_1)$. Assume (1) is valid for n. Observe that

$$(x_1', x_2', \cdots, x_{n+1}') \preceq (x_1, x_2, \cdots, x_{n+1})$$

if and only if

(i) $$s'_{n+1} < s_{n+1},$$

or

(ii) $$s'_{n+1} = s_{n+1} \text{ and } x'_{n+1} < x_{n+1},$$

or

(iii) $$s'_{n+1} = s_{n+1},\ x'_{n+1} = x_{n+1} \text{ and } (x'_1, \cdots, x'_n) \preceq (x_1, \cdots, x_n).$$

Thus if

$$\alpha = (x_1, x_2, \cdots, x_{n+1}),$$

M_α may be expressed as the union of three disjoint sets A, B and C which consist of those elements of N^{n+1} satisfying, respectively, conditions (i), (ii), and (iii). Thus,

$$f_{n+1}(\alpha) = \#(M_\alpha) = \#(A) + \#(B) + \#(C).$$

We now compute $\#(A) + \#(B) + \#(C)$. We will have occasion to use the combinatorial identity,

(2) $$\sum_{j=t+1}^{t+r} \binom{j-1}{t} = \binom{t+r}{t+1}$$

(which may be established by induction on r) and the fact that the number of n-tuples of positive integers which satisfy the equation $x_1 + \cdots + x_n = t$ is

$$\binom{t-1}{n-1}.$$

(Think of placing t objects in a row and placing dividers into $n - 1$ of the $t - 1$ spaces between the objects. Then x_1 is the number of objects before the first divider, x_2 is the number between the first and second dividers, etc.)

Note that $\beta = (y_1, y_2, \cdots, y_{n+1})$ is an element of A if and only if $y_1 + y_2 + \cdots + y_{n+1} = j$ where $n + 1 \le j < s_{n+1}$. Thus,

$$\#(A) = \sum_{j=n+1}^{s_{n+1}-1} \binom{j-1}{n} ,$$

and hence, using (2),

$$\#(A) = \binom{s_{n+1} - 1}{n+1} .$$

Now $\beta \in B$ if and only if $1 \le y_{n+1} \le x_{n+1} - 1$ and

$$y_1 + \cdots + y_{n+1} = x_1 + \cdots + x_{n+1} = s_{n+1} .$$

Thus $\beta \in B$ if and only if $y_1 + \cdots + y_n = j$ where $s_n + 1 \le j \le s_{n+1} - 1$. Hence,

$$\#(B) = \sum_{j=s_n+1}^{s_{n+1}-1} \binom{j-1}{n-1} .$$

Using (2), we have

$$\#(B) = \sum_{j=n}^{s_{n+1}-1} \binom{j-1}{n-1} - \sum_{j=n}^{s_n} \binom{j-1}{n-1} = \binom{s_{n+1}-1}{n} - \binom{s_n}{n} .$$

Finally, $\beta \in C$ if and only if

$$y_{n+1} = x_{n+1} ,$$

$$y_1 + \cdots + y_n = s_n ,$$

and

$$(y_1, \cdots, y_n) \preceq (x_1, \cdots, x_n) .$$

The least such β is the $(n + 1)$-tuple

$$(s_n - n + 1, 1, 1, \cdots, 1, x_{n+1}) .$$

Thus $\beta \in C$ if and only if

$$(s_n - n + 1, 1, \cdots, 1) \preceq (y_1, \cdots, y_n) \preceq (x_1, \cdots, x_n) .$$

Hence,

$$\#(C) = f_n(x_1, \cdots, x_n) - f_n(s_n - n + 1, 1, \cdots, 1) + 1 .$$

Therefore, using the induction hypothesis and (2), we have

$$\#(C) = \left[\binom{s_n}{n} - \sum_{k-1}^{n-1}\binom{s_k - 1}{k}\right] - \left[\binom{s_n}{n} - \sum_{k=1}^{n-1}\binom{s_n - n + k - 1}{k}\right] + 1$$

$$= -\sum_{k=1}^{n-1}\binom{s_k - 1}{k} + \sum_{k=s_n-n}^{s_{n-1}}\binom{k - 1}{s_n - n - 1}$$

$$= -\sum_{k=1}^{n}\binom{s_k - 1}{k} + \binom{s_n - 1}{n} + \binom{s_n - 1}{s_n - n} .$$

Thus, since

$$\binom{s_{n+1} - 1}{n + 1} + \binom{s_{n+1} - 1}{n} = \binom{s_{n+1}}{n + 1}$$

and

$$\binom{s_n - 1}{n} + \binom{s_n - 1}{s_n - n} = \binom{s_n}{n} ,$$

we have

$$f_{n+1}(x_1, \cdots, x_{n+1}) = \#(A) + \#(B) + \#(C) = \binom{s_{n+1}}{n + 1} - \sum_{k=1}^{n} \binom{s_k - 1}{k} ,$$

and the theorem is established.

3. Theorem 2. Define $g: \bigcup_{n \in N} N^n \to N$ by

$$g(x_1, \cdots, x_n) = 2^{s_n - 1} - 1 + \sum_{k=1}^{n} \binom{s_n - 1}{k - 1} - \sum_{k=1}^{n-1} \binom{s_k - 1}{k} ,$$

where

$$s_k = \sum_{i=1}^{k} x_i$$

for $k \le n$ and $\binom{m}{k}$ is defined to be 0 if $m < k$. Then g is a one-one correspondence.

Proof. Define a relation $\triangleleft$ on $\bigcup_{n \in N} N^n$ as follows:

Definition. $(x'_1, \cdots, x'_m) \triangleleft (x_1, \cdots, x_n)$ if and only if

(i) $$s'_m < s_n ,$$

or

(ii) $$s'_m = s_n \text{ and } m < n ,$$

or

(iii) $s'_m = s_n$, $m = n$ and $(x'_1, \cdots, x'_n) \prec (x_1, \cdots, x_n)$.

The relation $\lhd$ well-orders $\bigcup_{n\in N} N^n$. For $\alpha \in \cup N^n$, let

$$s_\alpha = \{\beta \in N^n \mid \beta \unlhd \alpha\},$$

and let $g(\alpha) = \#(S_\alpha)$. Then g is a one-one mapping from $\bigcup_{n\in N} N^n$ onto N. We may express S_α as the union of three disjoint sets X, Y, and Z which consist of those elements of $\bigcup_{n\in N} N^n$ satisfying, respectively, conditions (i), (ii), and (iii) in the definition of $\lhd$.

Now $\beta = (y_1, \cdots, y_m) \in X$ if and only if $y_1 + \cdots + y_m = j$ where $1 \le j \le s_n - 1$. The number of elements in $\bigcup_{n\in N} N^n$ satisfying this equation for fixed j is

$$\sum_{m\in N} \binom{j-1}{m-1} = \sum_{m=1}^{j} \binom{j-1}{m-1} = 2^{j-1} .$$

Thus

$$\#(X) = \sum_{j=1}^{s_n - 1} 2^{j-1} = 2^{s_n - 1} - 1 .$$

We have β Y if and only if $y_1 + \cdots + y_m = s_n$ where $m < n$. Thus

$$\#(Y) = \sum_{m=1}^{n-1} \binom{s_n - 1}{m - 1} .$$

Finally, $\beta \in Z$ if and only if $\beta \in N^n$ and $\beta_0 \preceq \beta \preceq (x_1, \cdots, x_n)$ where β_0 is the n-tuple $(s_n - n + 1, 1, 1, \cdots, 1)$. Thus, using the result of Theorem 1 and (2), we have

$$\#(Z) = f_n(x_1, \cdots, x_n) - f_n(s_n - n + 1, 1, \cdots, 1) + 1 =$$

$$= \binom{s_n}{n} - \sum_{k=1}^{n-1}\binom{s_k - 1}{k} - \left[\binom{s_n}{n} - \sum_{k=1}^{n-1}\binom{s_n - n + k - 1}{k}\right] + 1$$

$$= -\sum_{k=1}^{n-1}\binom{s_k - 1}{k} + \sum_{k=s_n-n}^{s_n-1}\binom{k - 1}{s_n - n - 1}$$

$$= -\sum_{k=1}^{n-1}\binom{s_k - 1}{k} + \binom{s_n - 1}{s_n - n} \quad .$$

Therefore,

$$g(x_1, \cdots, x_n) = \#(X) + \#(Y) + \#(Z) =$$

$$= 2^{s_n - 1} - 1 + \sum_{k=1}^{n-1}\binom{s_n - 1}{k - 1} - \sum_{k=1}^{n-1}\binom{s_k - 1}{k} + \binom{s_n - 1}{s_n - n}$$

$$= 2^{s_n - 1} - 1 + \sum_{k=1}^{n}\binom{s_n - 1}{k - 1} - \sum_{k=1}^{n-1}\binom{s_k - 1}{k} \quad .$$

SOME RESULTS IN TRIGONOMETRY

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Graphs of the six circular functions in the first quadrant yield some particularly elegant results involving the Golden Section.

Let $\varphi^2 + \varphi = 1$, so that $\varphi = (\sqrt{5} - 1)/2 = 0.61803$ and notice that:

$$\text{arc cos } \varphi = \text{arc sin}\sqrt{1 - \varphi^2} = \text{arc sin}\sqrt{\varphi} = 0.90459$$

$$\text{arc sin } \varphi = \text{arc cos}\sqrt{1 - \varphi^2} = \text{arc cos}\sqrt{\varphi} = 0.66621$$

Further, if $\tan x = \cos x$, then $\sin x = \cos^2 x$ and $\sin^2 x + \sin x = 1$, that is, $x = \text{arc sin}\,\varphi$ in which case $\tan \text{arc sin}\varphi = \cos \text{arc sin}\varphi = \cos \text{arc cos}\sqrt{\varphi} = \sqrt{\varphi}$

[Continued on p. 392.]