CERTAIN ARITHMETICAL PROPERTIES OF ½k(ak± 1)

JOSEPH ARKIN
Spring Valley, New York

Define

$$A_u = \tfrac{1}{2}u(au - 1) \qquad \text{and} \qquad B_u = \tfrac{1}{2}u(au + 1),$$

where $a \neq 0$ is a positive rational integer.

In this paper, we discuss

$$A_u + A_v = A_k, \quad B_u + B_v = B_k, \quad A_uA_v = A_k \quad \text{and} \quad B_uB_v = B_k,$$

where the suffixes (u, v, k) are positive rational integers. In particular, we shall, for the first time, finally settle the question, and prove that if one solution of

$$\tfrac{1}{2}u(au \pm 1)\tfrac{1}{2}v(v \pm 1) = \tfrac{1}{2}k(k \pm 1)$$

exists for integral u, v, k, then an infinite number of other such solutions also exist.

Theorem 1. If a is an odd integer, then the suffixes are integers in

(2) $$A_{\frac{1}{2}(a^2q^2+(2a-1)q+2)} = A_{aq+1} + A_{\frac{1}{2}(a^2q^2+2aq-q)},$$

and

$$B_{\frac{1}{2}(a^2q^2+(2a+1)q+2)} = B_{\frac{1}{2}(a^2q^2+2aq+q)} + B_{aq+1},$$

where $q = 0, 1, 2, \cdots$.

Proof. The proof is immediate, using elementary algebra to show identities.

Theorem 2. If

(3) $$n = \tfrac{1}{2}(a^4q^2 + (2a - 1)aq + 2), \qquad m = a^2q + 1,$$

and

$$w = (a(n^2 + 1) - (n - 1))/2a ,$$

then

$$A_n = A_{n-1} + A_m = A_w - A_{w-1}, \quad (\text{with } q = 0, 1, 2, \cdots) ,$$

where the suffixes are integers when a is an odd integer.

Proof. In (2), we replace q with aq and then solve for w in

$$A_n = \tfrac{1}{2}n(n - 1) = \tfrac{1}{2}(2aw - a - 1) = A_w - A_{w-1} .$$

We complete the proof by observing that w and n are integers when a is odd.

In the same way we got (3), we get the following:

Corollary. If

$$n = \tfrac{1}{2}(a^4q^2 + (2a + 1)aq + 2), \quad m = a^2q + 1, \quad \text{and} \quad w = (a(n^2 + 1) + n - 1)/2a ,$$

then

$$B_n = B_{n-1} + B_m = B_w - B_{w-1}, \quad (\text{with } q = 0, 1, 2, \cdots) ,$$

where the suffixes are integers when a is an odd integer.

Remark. It should be noted that R. T. Hansen, in a recent paper [1], found solutions for the special case when $a = 3$ in (2), for the A sum, and in (3).

We now discuss the paired products in the following:

$$A_uA_v = A_k \qquad \text{and} \qquad B_uB_v = B_k ,$$

for integer suffixes (u, v, k).

Theorem 3. If a is an odd integer, the Pell equation,

(4) $$K^2 = 8ap^2 + 8a + 1$$

is solvable in rational integers, and $K + 1$ and $2p^2 + 1 \equiv 0 \pmod{a}$, then in $A_v A_u = A_k$, the suffixes (u, v, k) are the following integers

(4.1)
$$k = (2p^2 + 1)(2p^3 + 2p^2 + 2p + 1)/a ,$$
$$u = (2p^2 + 1)(2p^2 + 2p + 1)/a,$$

and

$$v = (K + 1)/2a = (1 + (8ap^2 + 8a + 1)^{\frac{1}{2}})/2a .$$

Proof. It is evident (by elementary means) that the identities in (4.1) balance the equation $A_v A_u = A_k$. We complete the proof by noting that the congruences are self-evident in (4.1).

Theorem 4. If a is an odd integer, the Pell equation

(5) $$K = 8ap^2 + 8a + 1$$

is solvable in rational integers, and $K - 1$ and $4p^2 + 3 \equiv 0 \pmod{a}$, then in $B_u B_v = B_k$, the suffixes (u, v, k) are the following integers

(5.1)
$$k = (4p^2 + 3)(4p^5 + 4p^4 + 5p^3 + 3p^2 + p)/a ,$$
$$u = (4p^2 + 3)(4p^4 + 4p^3 + 3p^2 + p)/a,$$

and

$$v = (K - 1)/2a = (1 + (8ap^2 + 8a + 1)^{\frac{1}{2}})/2a .$$

Proof. It is evident (by elementary means) that the identities in (5.1) balance the equation $B_u B_v = B_k$. The congruences in (5.1) of a are self-evident.

Euler [2] proved that if

(6) $$y^2 - Ax^2 = B$$

is solvable in integers, its solution reduces to the integration of the equation

$$y_{t+2} - 2my_{t+1} = 0$$

in finite differences, the integral being

$$y = (r + s)/2, \qquad x = (r - s)/(2(A)^{\frac{1}{2}}),$$

where

$$r = (Y + X(A)^{\frac{1}{2}})(m + n(A)^{\frac{1}{2}})^{z-1}, \qquad s = (Y - X(A)^{\frac{1}{2}})(m - n(A)^{\frac{1}{2}})^{z-1},$$

Y, X being the least integral solutions of $Y^2 - AX^2 = B$, and m, n being the least integral solutions of $m^2 - An^2 = 1$. This is Euler's theorem in changed notation.

Theorem 5. If a is an odd prime,

(7) $$K_t,\ P_t \qquad (t = 1, 2, 3, \cdots)$$

are integer solutions (where K_1, P_1 are the least integer solutions) of

$$\phi(t) = K_t^2 = 8aP_t^2 + 8a + 1,$$

and if there exists a K_j and a P_j which are the least integer solutions of $\phi(t)$ such that $K_j + 1 \equiv 0 \pmod{a}$ and $2P_j^2 \equiv -1 \pmod{a}$, then the number of solutions of $A_uA_v = A_k$ are infinite for the following integer suffixes (u, v, k):

(7.1)
$$k = (2P_i^2 + 1)(2P_i^3 + 2P_i^2 + 2P_i + 1)/a,$$

$$u = (2P_i^2 + 1)(2P_i^2 + 2P_i + 1)/a,$$

and

$$v = (K_i + 1)/2a = (1 + (8aP_i^2 + 8a + 1)^{\frac{1}{2}})/2a,$$

where $i = j + w(a-1)a \quad (w = 0, 1, 2, \cdots)$.

Proof. Since K_j, P_j are integers, $K_j + 1 \equiv 0 \pmod a$ and $2P_j^2 + 1 \equiv 0 \pmod a$, then combining (7.1) with (4.1), it is evident that the u, v, k in (7.1) are integers.

Now, combining (6) with the equation $\phi(j)$ in (7), we write

$$(8) \quad \begin{aligned} (K_j + P_j(8a)^{\frac{1}{2}})(m + n(8a)^{\frac{1}{2}})^{wa(a-1)} &= K_{wa(a-1)+j} + P_{wa(a-1)+j}(8a)^{\frac{1}{2}}, \\ (K_j - P_j(8a)^{\frac{1}{2}})(m - n(8a)^{\frac{1}{2}})^{wa(a-1)} &= K_{wa(a-1)+j} - P_{wa(a-1)+j}(8a)^{\frac{1}{2}}, \end{aligned}$$

where a is an odd prime and $w = 0, 1, 2, \cdots$.

In (6), it is evident that $(m,a) = 1$, and since a is an odd prime, we have, by Fermat's familiar theorem (m, a are integers with a an odd prime, $(m,n) = 1$, then $m^{a-1} \equiv 1 \pmod a$)

$$(m \pm n(8a)^{\frac{1}{2}})^{wa(a-1)} \equiv 1 \pmod a,$$

which leads to (in (8)),

$$K_j \equiv K_{wa(a-1)+j} \pmod a \quad \text{and} \quad P_j \equiv P_{wa(a-1)+j},$$

and we complete the proof by noting that these congruences satisfy the conditions of Theorem 5.

Corollary 1. In (7), it is almost immediate that

$$(9) \qquad 1 \le j \le a(a-1)$$

Since, if $j = sa(a-1) + d$ (where $1 = d = a(a-1)$ and $a = 0, 1, 2, \cdots$), it is evident that

$$K_{wa(a-1)+j} = K_{a(a-1)(w+s)+d} \equiv K_d \pmod a,$$

and

$$P_{wa(a-1)+j} = P_{a(a-1)(w+s)+d} \equiv P_d \pmod a.$$

Corollary 2. If a is an odd prime, and

$$K_j - 1 \equiv 0 \pmod a \quad \text{and} \quad 4P_j^2 + 3 \equiv 0 \pmod a ,$$

then the number of solutions of $B_u B_v = B_k$ are infinite for the following integer suffixes (u, v, k):

$$k = (4P_1^2 + 3)(4P_1^5 + 4P_1^4 + 5P_1^3 + 3P_1^2 + P_1)/a,$$
$$u = (4P_1^2 + 3)(4P_1^4 + 4P_1^3 + 3P_1^2 + P_1)/a ,$$

and

$$v = (K_j - 1)/2a = (1 + (8aP_1^2 + 8a + 1)^{\frac{1}{2}})/2a ,$$

where

$$i = j + wa(a - 1) \qquad (w = 0, 1, 2, \cdots) ,$$

and

$$1 \leq j \leq a(a - 1) .$$

We shall give one application in pentagonal numbers for infinite paired products in (7-7.1).

In (7-7.1), let $a = 3$, then

$$K^2 = 24P^2 + 25 \quad \text{and} \quad m^2 = 24n^2 + 1,$$

where the first solutions are

$$K_1 = 7, \quad P_1 = 1, \quad \text{and} \quad m = 5, \quad n = 1 .$$

We then find that $j = 4$ and 6, so that $i = 6w + 4$ and $6w + 6$, and we write

$$(7 + (24)^{\frac{1}{2}})(5 + (24)^{\frac{1}{2}})^{6w+3} = (K_4 + P_4(24)^{\frac{1}{2}})(5 + (24)^{\frac{1}{2}})^{6w} = K_{6w+4} + P_{6w+4}(24)^{\frac{1}{2}} = r,$$

and

$$(7 - (24)^{\frac{1}{2}})(5 - (24)^{\frac{1}{2}})^{6w+3} = (K_4 - P_4(24)^{\frac{1}{2}})(5 - (24)^{\frac{1}{2}})^{6w} = K_{6w+4} - P_{6w+4}(24)^{\frac{1}{2}} = s,$$

so that $(r + s)/2 = K_{6w+4}$ and $(r - s)/2 = P_{6w+4}$, In the same way, we find $(r + s)/2 = K_{6w+6}$ and $(r - s)/2 = P_{6w+6}$. Then combining these results with (6) and (7.1), we conclude our application.

REFERENCES

1. R. T. Hansen, "Arithmetic of Pentagonal Numbers," Fibonacci Quarterly, Vol. 8, No. 2 (1970), pp. 83-87.
2. L. Euler, Comm. Arith. Coll., I, pp. 316-336.

[Continued from page 475.]

4). Hence, if odd prime p divides F_{2k-1}, then p is not of the form 4s + 3, thus proving Conjecture 2 of Dmitri Thoro.* The proof by Leonard Weinstein** came to my attention at a later time and is distinct from the above proof.

*Dmitri Thoro, "Two Fibonacci Conjectures," Fibonacci Quarterly, Oct. 1965, pp. 184-186.

** Leonard Weinstein, "Letter to the Editor," Fibonacci Quarterly, Feb. 1966, p. 88.

ERRATA

Please make the following corrections in "Some Results on Fibonacci Quaternions," Vol. 7, No. 2, pp. 201-210.

Page 201 — The first displayed equation on the page should read:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k; \quad jk = -kj = i; \quad ki = -ik = j.$$

Page 205 — Change the bracketed part of Eq. (27) to read:

$$[F_r^2T_0 + F_{2r}(Q_0 - 3k)] .$$

Page 208 — Change the first terms of Eq. (74) to read:

$$T_{n+t}F_{n+r} = \quad \cdots$$