DETERMINATION OF HERONIAN TRIANGLES

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1. A Pythagorean triangle is defined as any right-triangle having integral sides. Using the well-known relationship $a^2 + b^2 = c^2$ where a, b, and c are the two sides and the hypotenuse respectively, it is obvious that one of the sides must be even. Hence, the area of such a triangle is also an integer. In his book*, Ore introduces the generalization of this situation: a triangle is called Heronian if it has integral sides and area. He further comments that, "although we know a considerable number of Heronian triangles, we have no general formula giving them all." In this paper, we propose to find all such triangles and prove a few basic properties concerning them.

2. Since every Pythagorean triangle is Heronian, and since Pythagorean triangles are completely described by the well-known formulas for the sides $u^2 + v^2$, $u^2 - v^2$, and $2uv$, the real problem is to characterize all non-right-angled Heronian triangles. We first give an obvious property.

Lemma 1. Let a, b, c, and n all be integers. Then the triangle with sides of na, nb, and nc is Heronian if and only if the "reduced" triangle with sides a,b,c is Heronian.

Proof. We shall use Heron's formula for the area of a triangle

$$A = \sqrt{s(s-a)(s-b)(s-c)}, \tag{1}$$

where

$$s = \frac{1}{2}(a + b + c).$$

Let A be the area of triangle a,b,c and let A' be the area of na, nb, nc. Then Eq. (1) shows us immediately that

*Oystein Ore, Invitation to Number Theory, pp. 59-60, Random House, New York, 1967.

$$A' = n^2 A . \tag{2}$$

Hence, if A is an integer, so is A'. For the converse, suppose that A' is integral. Then Eq. (2) insures that A is at least rational. On the other hand, Eq. (1) implies that A is the square root of an integer, which is well known to be either integral or irrational. Thus we conclude that A must be an integer and the lemma is proven.

Before we proceed to our first theorem, we illustrate with two examples. Suppose we juxtapose (or "adjoin") the two Pythagorean triangles 5, 12, 13 and 9, 12, 15 so that their common-length sides coincide. Clearly a (non-right-angled) Heronian triangle results with sides of 13, 14, 15 and area equalling 84.

As a second example, we adjoin the triangles 12, 16, 20 and 16, 63, 65 (one of which is primative) to obtain the Heronian triangle 20, 65, 75 which may be reduced to 4, 13, 15, a primitive Heronian triangle with area equalling 24. That these two examples illustrate all possible events is the content of our first theorem.

Theorem 1. A triangle is Heronian if and only if it is the adjunction of two Pythagorean triangles along a common side, or a reduction of such an adjunction.

Proof. One direction is clear: since every Pythagorean triangle is Heronian, so is every adjunction of two. The previous lemma guarantees that every reduction is also Heronian.

Thus, let us suppose that the triangle a, b, c is Heronian and try to prove that it is either the adjunction or a reduction of an adjunction of two Pythagorean triangles. First we assume the obvious: that from some vertex we may draw a perpendicular to the opposite side, thus dividing the given triangle into two "adjoined" right triangles. That such is possible is easily shown. Let the length of the altitude so constructed be x and let the base c be thus divided into segments c_1 and c_2, so that $c = c_1 + c_2$. Since the triangle is Heronian, it is clear that

$$x = \frac{2A}{c}$$

is rational. Let

$$x = \frac{m}{n}$$

be reduced to lowest terms.

By the law of cosines, $b^2 = a^2 + c^2 - 2ac \cos w_1$ and thus

$$\cos w_1 = \frac{a^2 + c^2 - b^2}{2ac}$$

is rational. Hence both $c_1 = b \cos w_1$ and $c_2 = c - c_1$ are also rational. If the numbers x, c_1, and c_2 are in fact all integers, we are done. Otherwise we look at the triangle having sides na, nb, and nc. From elementary geometry, this triangle is similar to the original one and thus the new altitude is equal to $nx = m$, an integer. But nc_1, still rational, is given by

$$\sqrt{n^2a^2 - m^2},$$

which is, as before, either integral or irrational, and thus must be integral. Likewise nc_2 is integral, and the new enlarged triangle is the adjunction of two Pythagorean triangles. Thus the original is a reduction of an adjunction and we are through.

Since the sides of any Pythagorean triangle are given by $u^2 + v^2$, $u^2 - v^2$, and $2uv$, we now have a method for finding all Heronian triangles.

Corollary 1. A triangle is Heronian if and only if its sides are given by either (3) $u^2 + v^2$, $r^2 + s^2$, and $u^2 - v^2 + r^2 - s^2$; where $rs = uv$; or (4) $u^2 + v^2$, $r^2 + s^2$, and $2(uv + rs)$; where $r^2 - s^2 = u^2 - v^2$; or (5) a reduction by any constant factor in either case (3) or (4).

3. Although the preceding theorem and its corollary give formulations for finding all Heronian triangles, there are many properties of Heronian triangles that are not obvious from examination of the special subset of right-angled triangles. Some of these properties will be given here.

Lemma 2. A primitive Heronian triangle is isosceles if and only if it has sides given by (3), (4), or (5) with $r = u$ and $s = v$.

Proof. Since a triangle is primitive only when one side is even, the equal sides of the isosceles triangle must be odd, say $2m + 1$. Let the even side be $2n$. Then the semiperimeter is given by $2m + n + 1$ and the area, given by Eq. (1) becomes

$$\sqrt{(2m + n + 1)(2m - n + 1)(n)(n)}\ ,$$

so that

$$A = n\sqrt{(2m + 1)^2 - n^2}\ .$$

If this is to be an integer, there must be an integer Q such that

$$(2m + 1)^2 - n^2 = Q^2\ .$$

Thus the number $2m + 1$ is the hypotenuse of a Pythagorean triangle, which means, of course, that $2m + 1$ is as given in Corollary 1. Conversely, every triangle described by those formulae will be isosceles if $r = u$ and $s = v$.

We note in particular that any number of the form $4n + 2$ may be used as the even side of a primitive isosceles Heronian triangle, by using sides $2n^2 + 2n + 1$, $2n^2 + 2n + 1$, and $4n + 2$. We will show, in fact, that any integer greater than two may be used as the side of a primitive non-right-angled Heronian triangle. Before we do, we shall establish the following.

Lemma 3. No Heronian triangle has a side of either 1 or 2.

Proof. Since the sides of the triangle must be integers, the difference between two sides is either 0 or an integer ≥ 1. This latter case would preclude the use of 1 as a side. But an isosceles triangle with side one is also impossible, since in that event, we must have two sides equal to 1, and hence the third side either 0 or 2. Thus we have only to show that 2 cannot be used as the side of an Heronian triangle. Suppose, to the contrary, that we do have a triangle with sides 2, a, and b. Then the area as given by (1) is

$$A = \sqrt{s(s - 2)(s - a)(s - b)}\ ,$$

where $a + b + 2 = 2s$. The only values of a and b which satisfy this last equation are $a = b = s - 1$. Thus the area becomes

$$A = \sqrt{s(s-2)(1)(1)}$$

and we must have $s(s-2) = Q^2$ for some Q. But this is impossible, so we are done.

Using the formulae for the sides of a Pythagorean triangle, it is easy to show that every integer greater than two can be used as a side in a finite number of Pythagorean triangles. This observation has the following remarkable generalization.

Theorem 2. Let a be an integer greater than two. Then there exists an infinitude of primitive Heronian triangles with one side of length a.

Proof. If a is odd, we may use sides given by

$$a,\ \frac{1}{2}(at-1), \quad \text{and} \quad \frac{1}{2}(at+1) \tag{6}$$

where t is a solution of the Pellian equation

$$t^2 - (a^2-1)y^2 = 1 . \tag{7}$$

Since $a^2 - 1$ is even and never a perfect square, Eq. (7) has an infinitude of solutions for t, each of them odd, so that (6) lists only integers. Since

$$\frac{1}{2}(at-1) + 1 = \frac{1}{2}(at+1),$$

the triangle is obviously primitive. We compute the area of the triangle by (1) and find

$$A^2 = \frac{1}{2}(t+1)a\frac{1}{2}(t-1)a\frac{1}{2}(a+1)\frac{1}{2}(a-1) = (\tfrac{1}{4})^2a^2(a^2-1)(t^2-1) .$$

But, by Eq. (7), we have $t^2 - 1 = (a^2-1)y^2$ so that

$$A^2 = (\tfrac{1}{4})^2a^2(a^2-1)^2y^2$$

and thus

$$A = \frac{1}{2}(a-1)\frac{1}{2}(a+1)ay ,$$

an integer, and hence the triangle is Heronian.

Next, suppose that a is even, say $a = 2n$, where n is odd. Then we may use

$$(8) \qquad a, \quad tn - 2, \quad \text{and} \quad tn + 2,$$

where t is any odd solution of

$$(9) \qquad t^2 - (n^2 - 4)y^2 = 1.$$

Since n is odd, $n^2 - 4$ is also odd and thus an infinitude of odd values of t is available. That (8) forms a primitive triangle is clear; we prove it is Heronian by computing

$$A^2 = (t + 1)n(t - 1)n(n + 2)(n - 2) = n^2(n^2 - 4)(t^2 - 1) = n^2(n^2 - 4)^2(y)^2$$

so that A is an integer.

Lastly, suppose that $a = 2n$, where n is even. Then we may use

$$(10) \qquad a, \quad tn - 1, \quad \text{and} \quad tn + 1,$$

where t is any solution of

$$(11) \qquad t^2 - (n^2 - 1)y^2 = 1.$$

The proof follows the same lines as that just given. Thus our theorem is proven for all cases.

It should be obvious that the formulations given in the above proof are simply chosen from an infinitude of possibilities. Thus we could also have shown, for example, that the triangle having sides

$$a, \ \frac{1}{2}a(x - 1) + 1,$$

and

$$\frac{1}{2}a(x + 1) - 1,$$

where x is found from

$$x^2 - (a - 1)y^2 - 1,$$

which will have an infinitude of solutions as long as $a - 1$ is not a perfect square.

4. We conclude with a few observations and a short list of examples. It is easy to show that every primitive Heronian triangle has exactly one even side. A simple check of all the possibilities obtained from the adjunction of two Pythagorean triangles (which are known to have either one or three even sides) will suffice to prove this. From this, we conclude that the area of any Heronian triangle is divisible by 2, since, if s is even, a factor of 2 divides A^2 (and thus also A) while, if s is odd, then $s - a$ will be even where a is an odd side of the triangle, and so again 2 divides the area.

Since the area of any Pythagorean triangle is given by

$$A = uv(u - v)(u + v)$$

it is easy to show that such a triangle has area divisible by three. A simple analysis of adjunctions and possible reductions will then show that every Heronian triangle has area divisible by three.

Following is a list of the "first" few primitive (non-right-angled) Heronian triangles:

$a = 3$	$b = 25$	$c = 26$
4	13	15
5	5	6
5	5	8
5	29	30
6	25	29
7	15	20
8	29	35

a	b	c
a = 9	b = 10	c = 17
9	65	70
10	13	13
10	17	21
11	13	20
12	17	25
13	13	24
13	14	15
13	20	21
13	37	40
14	25	25
15	28	41
15	37	44
15	41	52
16	17	17
17	17	30
17	25	26
17	25	28
18	41	41
19	20	37
20	37	51
21	85	104
22	61	61
23	212	225
24	37	37
25	25	48
25	29	36
25	39	40
25	51	74
26	85	85
27	676	701
28	85	111
29	29	40
29	29	42

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