THE FIBONACCI NUMBERS CONSIDERED AS A PISOT SEQUENCE*

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Charles Pisot [1] was the first to consider the sequence, $\{a_n\}_{n=0}^{\infty}$, of natural numbers determined from two natural numbers a_0 and a_1 such that

$$2 \le a_0 < a_1$$

(1) and

$$-\frac{1}{2} < a_{n+2} - \frac{a_{n+1}^2}{a_n} \le \frac{1}{2}$$

for all $n \ge 0$. The Fibonacci numbers with the first two terms deleted satisfy Eq. (1).

Peter Flor [2] called the sequences which satisfy (1) Pisot sequences of the second kind. Flor also considered the sequence of natural numbers determined from two natural numbers a_0 and a_1 such that

$$2 \le a_0 < a_1$$

(2) and

$$-\frac{1}{2} \le a_{n+2} - \frac{a_{n+1}^2}{a_n} < \frac{1}{2}$$

for all $n \ge 0$. He called these sequences Pisot sequences of the first kind. For a Pisot sequence of the first (second) kind a_{n+2} is simply the nearest integer to a_{n+1}^2/a_n, where in case of ambiguity we choose the smaller

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(larger) integer. By Pisot sequence we shall mean a sequence that satisfies (1) and (2).

By

$$\{F_n + k\}_{n=n_0}^{\infty}$$

we mean the sequence formed by adding k to each term of the sequence

$$\{F_n\}_{n=n_0}^{\infty} ,$$

where F_n is the n^{th} Fibonacci number. In this paper necessary and sufficient conditions for

$$\{F_n + k\}_{n=n_0}^{\infty}$$

to be a Pisot sequence are given.

The main result is

Theorem. Let

$$\{F_n\}_{n=1}^{\infty}$$

be the Fibonacci sequence. The sequence

$$\{F_n + 1\}_{n=n_0}^{\infty}$$

is a Pisot sequence of the first kind (second kind) iff $n_0 \geq 6$ ($n_0 \geq 4$). The sequence

$$\{F_n - 1\}_{n=n_0}^{\infty}$$

is a Pisot sequence iff $n_0 \geq 7$. The sequence $\{F_n\}$ is a Pisot sequence of the first kind (second kind) iff $n_0 \geq 4$ ($n_0 \geq 3$). If $|k| > 1$ then there exists no integer n_0 such that

$$\{F_n + k\}_{n=n_0}^{\infty}$$

is a Pisot sequence.

We shall need two lemmas in order to prove the theorem.

Lemma 1. $F_{n+2} - 2F_{n+1} + F_n = F_{n-2}$.

Proof. $F_{n+2} - 2F_{n+1} + F_n = (F_{n+1} + F_n) - 2F_{n+1} + F_n$
$= -F_{n+1} + 2F_n = -(F_n + F_{n-1}) + 2F_n$
$= F_n - F_{n-1} = F_{n-2}$. ::

Lemma 2. $(F_{n+2} + k)(F_n + k) - (F_{n+1} + k)^2$
$= (-1)^{n+1} + kF_{n-2}$.

Proof. $(F_{n+2} + k)(F_n + k) - (F_{n+1} + k)^2$
$= (F_{n+2}F_n - F_{n+1}^2) + k(F_{n+2} - 2F_{n+1} + F_n)$
$= (-1)^{n+1} + kF_{n-2}$.

The last equality is true since

$$F_{n+2}F_n - F_{n+1}^2 = (-1)^{n+1} \quad ::$$

We are now able to begin the proof of the theorem. From the definition of a Pisot sequence and Lemma 2, we have that

$$\{F_n + k\}_{n=n_0}^{\infty}$$

is a Pisot sequence of the first kind iff

(i) $$2 \le F_{n_0} + k < F_{n_0+1} + k$$

and

(iia) $$-(F_n + k) \le 2[(-1)^{n+1} + kF_{n-2}] \qquad \text{for all } n \ge n_0$$

(iib) $$2[(-1)^{n+1} + kF_{n-2}] < F_n + k \qquad \text{for all } n \ge n_0 .$$

are satisfied. Also $\{F_n + k\}_{n=n_0}^{\infty}$ is a Pisot sequence of the second kind iff (i) and (iiia) $-(F_n + k) < 2[(-1)^{n+1} + kF_{n-2}]$ for all $n \geq n_0$

(iiib) $2[(-1)^{n+1} + kF_{n-2}] \leq F_n + k$ for all $n \geq n_0$.

We shall first consider the case $k = 1$.

$$F_n + 1 = 2F_{n-2} + F_{n-3} + 1 > 2[F_{n-2} + 1] \geq 2[F_{n-2} + (-1)^{n+1}]$$

iff $n \geq 6$. Thus (iib) is satisfied iff $n \geq 6$. Also,

$$F_n + 1 = 2F_{n-2} + F_{n-3} + 1 \geq 2[F_{n-2} + 1] \geq 2[F_{n-2} + (-1)^{n+1}]$$

iff $n \geq 4$. Thus (iiib) is satisfied iff $n \geq 4$. Since

$$2[(-1)^{n+1} + F_{n-2}] \geq 0 > -(F_n + 1) \qquad \text{for all } n \geq 3,$$

(iia) and (iiia) are satisfied for $n \geq 3$. It is clear that (i) is satisfied if $n_0 \geq 2$. Thus

$$\{F_n + 1\}_{n=n_0}^{\infty}$$

is a Pisot sequence of the first kind iff $n_0 \geq 6$ and it is a Pisot sequence of the second kind iff $n_0 \geq 4$.

Next, we consider the case $k = -1$. If $n = 6$, both (iia) and (iiia) are not satisfied. If $n = 7$, both (iia) and (iiia) are satisfied. Now

$$F_n - 1 = 2F_{n-2} + F_{n-3} - 1 > 2[F_{n-2} + 1] \qquad \text{if } n \geq 8 .$$

Thus,

$$-(F_n - 1) < 2[-1 - F_{n-2}] \leq 2[(-1)^{n+1} - F_{n-2}]$$

if $n \geq 8$. Therefore, (iia) and (iiia) are satisfied iff $n \geq 7$. Since

$$2[(-1)^{n+1} - F_{n-2}] \leq 0 < F_n - 1$$

if $n \geq 3$, both (iib) and (iiib) are satisfied for $n \geq 3$. It is clear that (i) is satisfied for $n \geq 4$. Thus,

$$\{F_n - 1\}_{n=n_0}^{\infty}$$

is a Pisot sequence iff $n_0 \geq 7$.

Now we consider the case $k = 0$. It is clear that (i) is satisfied iff $n \geq 3$. Both (iia) and (iiia) are satisfied for $n \geq 3$. Also (iiib) is satisfied for $n \geq 3$, but (iib) is satisfied iff $n \geq 4$. Thus

$$\{F_n\}_{n=n_0}^{\infty}$$

is a Pisot sequence of the first kind iff $n_0 \geq 4$, and

$$\{F_n\}_{n=n_0}^{\infty}$$

is a Pisot sequence of the second kind iff $n_0 \geq 3$.

We shall show that if $|k| > 1$, then there exists no integer n_0 such that

$$\{F_n + k\}_{n=n_0}^{\infty}$$

is a Pisot sequence. This will be accomplished by showing that (iia) or (iiia) implies that $k > -2$ and that (iib) or (iiib) implies that $2 < k$.

Dividing (iia) by F_{n-2} yields that

$$-\frac{F_n}{F_{n-1}} \cdot \frac{F_{n-1}}{F_{n-2}} - \frac{k}{F_{n-2}} \leq \frac{2(-1)^{n+1}}{F_{n-2}} + 2k$$

for $n \geq n_0$. After taking the limit of both sides as $n \to \infty$ and remembering that

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2} < 2 ,$$

we have that

$$-4 < -\lim \left(\frac{F_{n+1}}{F_n} \right)^2 \leq 2k .$$

Thus

$$-2 < k .$$

In a similar manner, one can show that (iib) or (iiib) implies that $k < 2$. ::

REFERENCES

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