ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

B-196 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let $a_0, a_1, a_2, \cdots$, and $b_0, b_1, b_2, \cdots$ be two sequences such that

$$b_n = \binom{n}{0}a_n + \binom{n}{1}a_{n-1} + \binom{n}{2}a_{n-2} + \cdots + \binom{n}{n}a_0 \qquad n = 0, 1, 2, \cdots$$

Give the formula for a_n in terms of $b_n, \cdots, b_0$.

B-197 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let the Pell Sequence be defined by $P_0 = 0$, $P_1 = 1$, and $P_{n+2} = 2P_{n+1} + P_n$. Show that there is a sequence Q_n such that

$$P_{n+2k} = Q_k P_{n+k} - (-1)^k P_n ,$$

and give initial conditions and the recursion formula for Q_n.

B-198 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let c_n be the coefficient of $x_1 x_2 \cdots x_n$ in the expansion of

$$(-x_1 + x_2 + x_3 + \cdots + x_n)(x_1 - x_2 + x_3 + \cdots + x_n)(x_1 + x_2 - x_3 + \cdots + x_n) \cdots (x_1 + x_2 + x_3 + \cdots + x_{n-1} - x_n) .$$

For example, $c_1 = -1$, $c_2 = 2$, $c_3 = -2$, $c_4 = 8$, and $c_5 = 8$. Show that

$$c_{n+2} = nc_{n+1} + 2(n+1)c_n, \qquad c_n = nc_{n-1} + (-2)^n,$$

and

$$\lim_{n \to \infty} (c_n/n!) = e^{-2} .$$

B-199 Proposed by M. J. DeLeon, Florida Atlantic University, Boca Raton, Florida.

Define the Fibonacci and Pell numbers by

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \qquad n \ge 1;$$
$$P_1 = 1, \quad P_2 = 2, \quad P_{n+2} = 2P_{n+1} + P_n \qquad n \ge 1 .$$

Prove or disprove that $P_{6k} < F_{11k}$ for $k \ge 1$.

B-200 Proposed by M. J. DeLeon, Florida Atlantic University, Boca Raton, Florida.

With the notation of B-199, prove or disprove that

$$F_{11k} < P_{6k+1} \quad \text{for } k \ge 1 .$$

B-201 Proposed by Mel Most, Ridgefield Park, New Jersey.

Given that a very large positive integer k is a term F_n in the Fibonacci Sequence, describe an operation on k that will indicate whether n is even or odd.

SOLUTIONS

DOUBLING NEED NOT BE TROUBLING

B-178 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida.

For all positive integers n show that

$$F_{2n+2} = \sum_{i=1}^{n} 2^{n-i} F_{2i-1} + z^n ,$$

and

$$F_{2n+3} = \sum_{i=1}^{n} 2^{n-i} F_{2i} + 2^{n+1} .$$

Generalize.

Solution by Herta T. Freitag, Hollins, Virginia.

Our generalization states that for all positive integers n and for all positive integers a

$$\sum_{i=1}^{n} 2^{n-1} F_{2i+(a-3)} + 2^n F_n = F_{2n+a} . \tag{1}$$

The proof is by mathematical induction on n.

Relationship (1), for $n = 1$, claims that

$$F_{a-1} + 2F_a = F_{a+2} ,$$

which is, indeed, the case

Assuming that (1) holds for some positive integer, say k, we have:

(2) $$\sum_{i=1}^{k} 2^{k-i} F_{2i+(a-3)} + 2^k F_a = F_{2k+a} \quad .$$

To see if

(3) $$\sum_{i=1}^{k+1} 2^{k+1-i} F_{2i+(a-3)} + 2^{k+1} F_n = F_{2k+a+2} \quad ,$$

we recognize—on the basis of our assumption (2)—that the left side of (3) equals

$$2F_{2k+a} + F_{2k+a-1} \quad ,$$

which, however, is seen to be the number $F_{2k+a} + F_{2k+a+1}$ and, hence, F_{2k+a+2}.

This completes our proof by the principle of mathematical induction.

The relationships stated in the problem now become special ases of our generalization (1), whereby $a = 2$ establishes the first, and $a = 3$ the second, of the two given formulas.

Also solved by C. B. A. Peck, A. G. Shannon (T.P.N.G.), and the Proposer.

A SURJECTION (NOT MONOTONIC)

B-179 Based on Douglas Lind's Problem B-165.

Let Z^+ consist of the positive integers and let the function b from Z^+ to Z^+ be defined by $b(1) = b(2) = 1$, $b(2k) = b(k)$, and $b(2k + 1) = b(k + 1) + b(k)$ for $k = 1, 2, \cdots$. Show that every positive integer m is a value of b(n) and that $b(n + 1) \geq b(n)$ for all positive integers n.

Solution

Put

$$B(x) = \sum_{k=1}^{\infty} b(k)x^{k-1} .$$

Then

$$B(x) = \sum_{k=1}^{\infty} b(2k)x^{2k-1} + \sum_{k=1}^{\infty} b(2k - 1)x^{2k-2}$$

$$= \sum_{k=1}^{\infty} b(k)\, x^{2k-1} + 1 + \sum_{k=1}^{\infty} [b(k) + b(k + 1)]x^{2k}$$

$$= xB(x^2) + x^2B(x^2) + \sum_{k=0}^{\infty} b(k + 1)x^{2k} ,$$

so that

$$B(x) = (1 + x + x^2)B(x^2) .$$

It follows that

$$B(x) = \prod_{n=0}^{\infty} (1 + x^{2^n} + x^{2^{n+1}}) .$$

It is evident from this generating function that every positive integer m is a value of $b(n)$. However, the statement $b(n + 1) \geq b(n)$ is false:

$$b(2k + 2) = b(k + 1) < b(k + 1) + b(k) = b(2k + 1) .$$

For additional properties of $b(k)$, see: "A Problem in Partitions Related to the Stirling Numbers," Bull. Amer. Math. Soc., Vol. 70 (1964), pp. 275-278; also: D. A. Lind, "An Extension of Stern's Diatomic Series," Duke Math. Journal, Vol. 36 (1969), pp. 53-60.

Outline

Every positive integer m is a value of b(n) since

$$m = b(2^{m-1} + 1).$$

This is easily established by mathematical induction using the definition of b(n) and the fact that $b(2^n) = 1$.

BUNNY PATHS?

B-180 Proposed by Reuben C. Drake, North Carolina A & T University, Greensboro, North Carolina.

Enumerate the paths in the Cartesian plane from (0,0) to (n,0) that consist of directed line segments of the four following types:

Type	I	II	III	IV
Initial Point	(k,0)	(k,0)	(k,1)	(k,1)
Terminal Point	(k,1)	(k + 1,0)	(k + 1,1)	(k + 1,0)

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Let f(n) denote the total number of paths from (0,0) to (n,0). Let $f_0(n)$ denote the number of paths ending with segment of Type II, and $f_1(n)$ the number ending with a segment of Type IV. Then we have

$$f_0(n + 1) = f_0(n) + f_1(n) = f(n)$$

$$f_1(n + 1) = f(0) + f(1) + \cdots + f(n).$$

It follows that

$$f(n + 1) = f(n) + \sum_{k=0}^{n} f(k).$$

Put

$$F(x) = \sum_{n=0}^{\infty} f(n)x^n .$$

Then

$$F(x) = 1 + \sum_{n=0}^{\infty} \left\{ f(n) + \sum_{k=0}^{n} f(k) \right\} x^n$$

$$= 1 + x\,F(x) + \frac{x}{1-x} F(x) ,$$

so that

$$F(x) = \frac{1-x}{1-3x+x^2} .$$

Since

$$\frac{x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n} x^n$$

in the usual notation for Fibonacci numbers, it follows that

$$f(n) = F_{2n+1} .$$

Moreover,

$$f_0(n) = F_{2n-1}$$

and

$$f_1(n) = F_1 + F_3 + \cdots + F_{2n-1} = F_{2n} .$$

Also solved by G. J. Giaccai and the Proposer.

AN INFINITE MATRIX

B-181 Proposed by J. B. Roberts, Reed College, Portland, Oregon.

Let m be a fixed integer and let $G_{-1} = 0$, $G_1 = 1$, $G_n = G_{n-1} + G_{n-2}$ for $n \geq 1$. Show that $G_0, G_m, G_{2m}, G_{3m}, \cdots$ is the sequence of upper left principal minors of the infinite matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots \\ G_{m-2} & G_{m-2} + G_m & 1 & 0 & \cdots \\ 0 & (-1)^m & G_{m-2} + G_m & 1 & \cdots \\ 0 & 0 & (-1)^m & G_{m-2} + G_m & \cdots \\ 0 & 0 & 0 & (-1)^m & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Solution by the Proposer.

Expansion of the typical minor M_k, $k > 2$, by means of the elements of its last row, yields the recurrence relation

$$M_k = (G_{m-2} + G_m)M_{k-1} - (-1)^m M_{k-2} .$$

Induction, making use of the identity

$$G_n = (G_{k-2} + G_k)G_{n-k} - (-1)^k G_{n-2k}$$

(itself easily proved by induction), yields the conclusion.

CONGRUENCES

B-182 Proposed by James E. Desmond, Florida State University, Tallahassee, Florida.

Show that for any prime p and any integer n,

$$F_{np} \equiv F_n F_p \pmod{p} \qquad \text{and} \qquad L_{np} \equiv L_n L_p \equiv L_n \pmod{p} .$$

Solution by the Proposer.

We have, from Hardy and Wright, Theory of Numbers, Oxford University Press, London, 1954, p. 150, that $F_{p-1} \equiv 0 \pmod p$ and $F_p \equiv 1 \pmod p$ if $p \equiv \pm 1 \pmod 5$ and that $F_{p+1} \equiv 0 \pmod p$ and $F_p \equiv -1 \pmod p$ if $p \equiv \pm 2 \pmod 5$. Therefore, $L_p \equiv 1 \pmod p$ for all primes p, the case $p = 5$ being clear.

From I. D. Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, Vol. 1, No. 2 (1963), p. 77, we have

$$F_{r+s} = L_s F_r + (-1)^{s+1} F_{r-s}$$

for all integers r and s. Let $s = p$, a prime, and let $r = np$ for an arbitrary integer n. Then

$$F_{(n+1)p} = L_p F_{np} + (-1)^{p+1} F_{(n-1)p}$$

so that

$$F_{(n+1)p} \equiv F_{np} + F_{(n-1)p} \pmod p$$

for any integer n and any prime p, the case $p = 2$ being clear. Similarly, we obtain

$$L_{(n+1)p} \equiv L_{np} + L_{(n-1)p} \pmod p$$

for any integer n and any prime p, since $L_m = F_{m+1} + F_{m-1}$ for all integers m. Now, using induction on n, the proposition is true for $n = 0, 1$. Suppose $F_{np} \equiv F_n F_p \pmod p$ for $n = 0, 1, \ldots, k$ with $k > 0$. Then

$$F_{(k+1)p} \equiv F_{kp} + F_{(k-1)p} \equiv F_k F_p + F_{k-1} F_p \pmod p .$$

Suppose $F_{np} \equiv F_n F_p \pmod p$ for $n = 1, 0, \cdots, k$ with $k > 1$. Then

$$F_{(k-1)p} \equiv F_{(k+1)p} - F_{kp} \equiv F_{k+1} F_p - F_k F_p \equiv F_{k-1} F_p \pmod p .$$

Therefore, the Fibonacci congruence relation is true for any prime p and any integer n. The Lucas congruence relation can be proved by an argument similar to that given above.

PALINDROME CUBES

B-183 Proposed by Gustavus J. Simmons, Sandia Corporation, Albuquerque, New Mexico.

A positive integer is a palindrome if its digits read the same forward or backward. The least positive integer n, such that n^2 is a palindrome but n is not, is 26. Let S be the set of n such that n^3 is a palindrome but n is not. Is S empty, finite, or infinite?

Comment by the Proposer.

Since 2201^3 is the palindrome 10662526601, S is not empty. This is all that is known about the set S.

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a	b	c
29	35	48
30	113	113
31	97	120
32	65	65
33	34	65
34	145	145
35	73	102
36	61	65
37	37	70
38	181	181
39	41	50
40	101	101