FIBONACCI NUMBERS AS PATHS OF A ROOK ON A CHESSBOARD

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The purpose of this article is to show that Fibonacci numbers can be derived by enumerating the number of different routes of a rook from one corner of a chessboard to the opposite corner when the moves of the rook are limited by restrictive fences.

Consider the chessboard array of binomial coefficients or figurate numbers in Fig. 1. It is well known that the number in any square or cell represents the number of different routes of a rook from the upper left corner to that cell, provided that the rook moves are either horizontal to the right or vertically downward.*

| | | 1 | ····· | | | | |
|---|---|----|-------|-----|-----|------|------|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 |
| 1 | 4 | 10 | 20 | 35 | 56 | 84 | 120 |
| 1 | 5 | 15 | 35 | 70 | 126 | 210 | 330 |
| 1 | 6 | 21 | 56 | 126 | 252 | 462 | 792 |
| 1 | 7 | 28 | 84 | 210 | 462 | 924 | 1716 |
| 1 | 8 | 36 | 120 | 330 | 792 | 1716 | 3432 |

Fig. 1. Number of Rook Paths from Corner of Chessboard

Figure 2 shows the same chessboard array in standard combinatorial notation:

$$\begin{pmatrix} h \\ k \end{pmatrix} = h!/k! (h - k)! = \begin{pmatrix} h \\ h - k \end{pmatrix} .$$

^{*}Edouard Lucas, Théorie des Nombres, Paris, 1891, page 83.

| . 19 | 70 | FI O | BONÀCC F A ROC | CI NUMI OK ON A | BERS A | S PATH SBOARI | IS) | |
|------|--|---|---------------------------------------|---------------------------------------|---------------------------------------|---------------------------------------|--|--|
| | $\begin{pmatrix} 0\\ 0 \end{pmatrix}$ | $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ | $\begin{pmatrix} 2\\ 0 \end{pmatrix}$ | $\begin{pmatrix} 3\\ 0 \end{pmatrix}$ | $\begin{pmatrix} 4\\ 0 \end{pmatrix}$ | $\begin{pmatrix} 5\\0 \end{pmatrix}$ | $\begin{pmatrix} 6\\0 \end{pmatrix}$ | · (7 |
| | $\begin{pmatrix} 1\\ 1 \end{pmatrix}$ | $\binom{2}{1}$ | $\begin{pmatrix} 3\\1 \end{pmatrix}$ | $\begin{pmatrix} 4\\1 \end{pmatrix}$ | $\begin{pmatrix} 5\\1 \end{pmatrix}$ | $\begin{pmatrix} 6\\1 \end{pmatrix}$ | $\binom{7}{1}$ | |
| | $\begin{pmatrix} 2\\2 \end{pmatrix}$ | $\begin{pmatrix} 3\\2 \end{pmatrix}$ | $\begin{pmatrix} 4\\2 \end{pmatrix}$ | $\begin{pmatrix} 5\\2 \end{pmatrix}$ | $\begin{pmatrix} 6\\2 \end{pmatrix}$ | $\begin{pmatrix} 7\\2 \end{pmatrix}$ | $\binom{8}{2}$ | $\binom{9}{2}$ |
| | $\begin{pmatrix} 3\\ 3 \end{pmatrix}$ | $\begin{pmatrix} 4\\ 3 \end{pmatrix}$ | $\begin{pmatrix} 5\\ 3 \end{pmatrix}$ | $\begin{pmatrix} 6\\ 3 \end{pmatrix}$ | $\begin{pmatrix} 7\\ 3 \end{pmatrix}$ | $\binom{8}{3}$ | $\begin{pmatrix} 9\\ 3 \end{pmatrix}$ | $\binom{10}{3}$ |
| | $\begin{pmatrix} 4\\4 \end{pmatrix}$ | $\binom{5}{4}$ | $\binom{6}{4}$ | $\begin{pmatrix} 7\\4 \end{pmatrix}$ | $\binom{8}{4}$ | $\binom{9}{4}$ | $\begin{pmatrix} 10\\ 4 \end{pmatrix}$ | $\begin{pmatrix} 11\\ 4 \end{pmatrix}$ |
| | $\binom{5}{5}$ | $\begin{pmatrix} 6\\5 \end{pmatrix}$ | $\begin{pmatrix} 7\\5 \end{pmatrix}$ | $\binom{8}{5}$ | $\begin{pmatrix} 9\\5 \end{pmatrix}$ | $\begin{pmatrix} 10\\5 \end{pmatrix}$ | $\binom{11}{5}$ | $\binom{12}{5}$ |
| | $\begin{pmatrix} 6\\6 \end{pmatrix}$ | $\begin{pmatrix} 7 \\ 6 \end{pmatrix}$ | $\binom{8}{6}$ | $\begin{pmatrix} 9\\6 \end{pmatrix}$ | $\begin{pmatrix} 10\\6 \end{pmatrix}$ | $\begin{pmatrix} 11\\6 \end{pmatrix}$ | $\begin{pmatrix} 12\\6 \end{pmatrix}$ | $\begin{pmatrix} 13\\7 \end{pmatrix}$ |
| | $\begin{pmatrix} 7 \\ 7 \end{pmatrix}$ | $\binom{8}{7}$ | $\begin{pmatrix} 9\\7 \end{pmatrix}$ | $\begin{pmatrix} 10\\7 \end{pmatrix}$ | $\begin{pmatrix} 11\\7 \end{pmatrix}$ | $\begin{pmatrix} 12\\7 \end{pmatrix}$ | $\begin{pmatrix} 13\\7 \end{pmatrix}$ | $\begin{pmatrix} 14\\ 7 \end{pmatrix}$ |

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Fig. 2. Rook Paths in Combinatorial Notation

Figure 3 shows a chessboard array where the moves of a rook are limited by the indicated pattern of horizontal and vertical restrictive fences.

| 1 | 1 | 1 | | | | | |
|---|---|---|----|----|----|-----|-------------|
| 1 | 2 | 3 | 3 | | | | |
| | 2 | 5 | 8 | 8 | | | |
| | | 5 | 13 | 21 | 21 | | |
| | | | 13 | 34 | 55 | 55 | |
| | | | | 34 | 89 | 144 | 1 44 |
| | | | | | 89 | 233 | 377 |
| | | | | | | 233 | 610 |

Fig. 3 Rook Paths Limited by Restrictive Fences

The array begins with number one in the top left corner. Inasmuch as the number in any cell is the sum of the numbers immediately above it and to the left of it, the pattern of restrictive fences results in the entire array being composed of Fibonacci numbers.

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Figure 4 shows the chessboard with the same pattern of fences as in Fig. 3, but with the numbers in the Fibonacci notation where $F_0 = F_1 = 1$; $F_2 = 2$; $F_3 = 3$; and, in general, $F_{n+2} = F_{n+1} + F_n$.

| And in case of the local division of the loc | | and the second second second | | | | | |
|--|----------------|------------------------------|------------------|----------------|-----------------|-----------------|-----------------|
| \mathbf{F}_{0} | F ₁ | F ₁ | | | | | |
| F ₀ | \mathbf{F}_2 | F_3 | F_3 | | | | |
| | \mathbf{F}_2 | $\mathbf{F_4}$ | \mathbf{F}_{5} | F_5 | | | |
| | | F_4 | F ₆ | F ₇ | F ₇ | | |
| | | | \mathbf{F}_{6} | F ₈ | F9 | F9 | |
| | | | | F ₈ | F ₁₀ | F ₁₁ | F11 |
| | | | | | F ₁₀ | F ₁₂ | F ₁₃ |
| | | | | | | F ₁₂ | F ₁₄ |

Fig. 4. Limited Rook Paths in Fibonacci Notation

Comparing Fig. 4 with Fig. 2, it is noted that F_0 corresponds to $\begin{pmatrix} 0\\0 \end{pmatrix}$ and F_{14} corresponds to $\begin{pmatrix} 14\\7 \end{pmatrix}$. Comparing Fig. 3 with Fig. 1, we see that the number of Fibonacci rook paths from corner to corner is 610, whereas the number of unrestricted paths is 3432. The difference of 2822 must be the number of routes which are eliminated because of the restrictive fences. This can be verified by tabulating the effect of each restrictive fence as in the analysis on the following page.

To generalize, in a chessboard of $(n + 1)^2$ cells, the number of unrestricted rook paths from corner to corner is $\binom{2n}{n}$, the number of Fibonacci rook paths is F_{2n} , and the number of paths that are eliminated by the pattern of horizontal and vertical fences is

$$\begin{pmatrix} 2n \\ n \end{pmatrix} - F_{2n} = F_0 \begin{pmatrix} 2n - 2 \\ n \end{pmatrix} + F_1 \begin{pmatrix} 2n - 3 \\ n \end{pmatrix} \\ + F_2 \begin{pmatrix} 2n - 4 \\ n - 1 \end{pmatrix} + F_3 \begin{pmatrix} 2n - 5 \\ n - 1 \end{pmatrix} \\ + \cdots \\ + F_{2n-6} \begin{pmatrix} 4 \\ 3 \end{pmatrix} + F_{2n-5} \begin{pmatrix} 3 \\ 3 \end{pmatrix} \\ + F_{2n-4} \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

| A | В | A x B |
|--|--|--|
| Number of Fibonacci paths from origin to cells with fences | Number of unrestricted rook paths from fence to lower right corner | Number of paths eliminated by fences |
| Cells with horizontal fences | | |
| $F_0 = 1$ | $\begin{pmatrix} 12\\7 \end{pmatrix}$ = 792 | 792 |
| $F_2 = 2$ | $\begin{pmatrix} 10\\6 \end{pmatrix} = 210$ | 420 |
| $F_4 = 5$ | $\binom{8}{5} = 56$ | 280 |
| $F_{6} = 13$ | $\begin{pmatrix} 6\\4 \end{pmatrix}$ = 15 | 195 |
| $F_8 = 34$ | $\begin{pmatrix} 4\\3 \end{pmatrix} = 4$ | 136 |
| $F_{10} = 89$ | $\begin{pmatrix} 2\\2 \end{pmatrix}$ = 1 | 89 |
| | Sub | total 1912 |
| Cells with vertical fences | | |
| $F_1 = 1$ | $\begin{pmatrix} 11\\7 \end{pmatrix} = 792$ | 330 |
| $F_3 = 3$ | $\begin{pmatrix} 9\\6 \end{pmatrix}$ = 84 | 262 |
| $\mathbf{F}_5 = 8$ | $\begin{pmatrix} 7\\5 \end{pmatrix} = 21$ | 168 |
| $F_7 = 21$ | $\begin{pmatrix} 5\\4 \end{pmatrix} = 5$ | 105 |
| $F_9 = 55$ | $\begin{pmatrix} 3\\3 \end{pmatrix} = 1$ | _55 |
| | Sub | total <u>910</u> |
| Total numb | 2822 | |
| Total numb | 4 610 | |
| Total numb | $\begin{pmatrix} 14\\7 \end{pmatrix}$ $\underline{3432}$ | |

ANALYSIS OF ELIMINATED ROOK PATHS

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