A FIBONACCI CIRCULANT

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1. Put

$$D_{n,r} = \begin{vmatrix} F_r & F_{r+1} & F_{r+2} & \cdots & F_{r+n-1} \\ F_{r+n-1} & F_r & F_{r+1} & \cdots & F_{r+n-2} \\ F_{r+n-2} & F_{r+n-1} & F_r & \cdots & F_{r+n-3} \\ \vdots & & & & \vdots \\ F_{r+1} & F_{r+2} & F_{r+3} & \cdots & F_r \end{vmatrix},$$

where F_n denotes the Fibonacci numbers defined by

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n .$$

We show that

$$D_{n,r} = \frac{(F_r - F_{n+r})^n - (F_{n+r-1} - F_{r-1})^n}{1 - L_n + (-1)^n}, \tag{1}$$

where $L_n = F_{n-1} + F_{n+1}$ is the n^{th} Lucas number.

A circulant is a determinant of the form

$$C(a_0, \cdots, a_{n-1}) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & & & & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{vmatrix} \tag{2}$$

It is known (see [1, Vol. 3, pp. 374-375] and [3, p. 39]) that

$$\text{(3)} \qquad C(a_0, \cdots, a_{n-1}) = \prod_{k=0}^{n-1} \left(\sum_{j=0}^{n-1} a_j \omega_k^j \right) ,$$

where the

$$\omega_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$$

are the n^{th} roots of unity. To establish (3) rapidly, multiply

$$C \equiv C(a_0, \cdots, a_{n-1})$$

by the Vandermonde determinant $V = \left|\omega_i^j\right|$ $(i, j = 0, 1, \cdots, n - 1)$. Denoting the right side of (3) by P, by factoring out common factors, one finds $CV = PV$, and since $V \neq 0$, (3) follows.

Now $D_{n,r}$ is a special case of (2) with

$$a_j = F_{j+r} = \frac{\alpha^{j+r} - \beta^{j+r}}{\alpha - \beta} ,$$

in which $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$. Thus by (3),

$$\begin{aligned} D_{n,r} &= \prod_{k=0}^{n-1} \left(\sum_{j=0}^{n-1} \frac{\alpha^r(\alpha\omega_k)^j - \beta^r(\beta\omega_k)^j}{\alpha - \beta} \right) \\ &= (\alpha - \beta)^{-n} \prod_{k=0}^{n-1} \left(\frac{\alpha^r[1 - (\alpha\omega_k)^n]}{1 - \alpha\omega_k} - \frac{\beta^r[1 - (\beta\omega_k)^n]}{1 - \beta\omega_k} \right) \\ &= \prod_{k=0}^{n-1} \frac{\alpha^r - \beta^r - \alpha^{n+r} + \beta^{n+r} + [\alpha^{r-1} - \beta^{r-1} - \alpha^{n+r-1} + \beta^{n+r-1}]\omega_k}{(\alpha - \beta)(1 - \alpha\omega_k)(1 - \beta\omega_k)} \\ &= \prod_{k=0}^{n-1} \frac{F_r - F_{n+r} - (F_{n+r-1} - F_{r-1})\omega_k}{(1 - \alpha\omega_k)(1 - \beta\omega_k)} . \end{aligned}$$

Now for any x and y,

$$(4) \quad \prod_{k=0}^{n-1} (x - y\omega_k) = y^n \prod_{k=0}^{n-1} \left(\frac{x}{y} - \omega_k\right) = y^n \left[\left(\frac{x}{y}\right)^n - 1\right] = x^n - y^n .$$

Therefore

$$\prod_{k=0}^{n-1} [F_r - F_{n+r} - (F_{n+r-1} - F_{r-1})\omega_k] = (F_r - F_{n+r})^n - (F_{n+r-1} - F_{r-1})^n ,$$

and

$$\prod_{k=0}^{n-1} (1 - \alpha\omega_k)(1 - \beta\omega_k) = \prod_{k=0}^{n-1} (1 - \alpha\omega_k) \prod_{k=0}^{n-1} (1 - \beta\omega_k)$$

$$= (1 - \alpha^n)(1 - \beta^n) = 1 - L_n + (-1)^n ,$$

where we have used $L_n = \alpha^n + \beta^n$. This establishes (1).

We note that this evaluation of $D_{n,k}$ simplifies if n is even. Ruggles [2] has shown that

$$F_{n+p} - F_{n-p} = \begin{cases} L_n F_p , & p \text{ even} \\ F_n L_p , & p \text{ odd} \end{cases} .$$

It follows that if $n \equiv 0 \pmod 4$,

$$D_{n,r} = \frac{F_{\frac{1}{2}n}^n \left[L_{r+\frac{1}{2}n}^n - L_{r-1+\frac{1}{2}n}^n\right]}{2 - L_n}$$

and if $n \equiv 2 \pmod 4$,

$$D_{n,r} = \frac{L_{\frac{1}{2}n}^n \left[F_{r+\frac{1}{2}n}^n - F_{r-1+\frac{1}{2}n}^n\right]}{2 - L_n} .$$

2. The generalization of (1) to second-order recurring sequences uses the same techniques. Consider the sequence $\{W_n\}$ defined by

$$W_{n+2} = pW_{n+1} - qW_n ,$$

W_0 and W_1 arbitrary, where $p^2 - 4q \neq 0$. Let a and b be the roots of the auxiliary polynomial, so that $a \neq b$ and $ab = q$. We shall assume that neither a nor b is an n^{th} root of unity. Since the roots are distinct, there are constants A and B such that $W_n = Aa^n + Bb^n$. Define the sequence $\{V_n\}$ by $V_n = a^n + b^n$.

Put

$$D_{n,r}(W) = C(W_r, W_{r+1}, \cdots, W_{r+n-1}) .$$

Setting $a_j = W_{j+r} = Aa^{j+r} + Bb^{j+r}$ in (3) gives

$$\begin{aligned} D_{n,r}(W) &= \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} Aa^r(a\omega_k)^j + Bb^r(b\omega_k)^j\right) \\ &= \prod_{k=0}^{n-1}\left(\frac{Aa^r(1 - a^n)}{1 - a\omega_k} + \frac{Bb^r(1 - b^n)}{1 - b\omega_k}\right) \\ &= \prod_{k=0}^{n-1} \frac{W_r - W_{n+r} - q(W_{r-1} - W_{n+r-1})\omega_k}{(1 - a\omega_k)(1 - b\omega_k)} \\ &= \frac{(W_r - W_{n+r})^n - q^n(W_{r-1} - W_{n+r-1})^n}{1 - V_n + q^n} , \end{aligned}$$

which agrees with (1) by taking $p = 1$, $q = -1$, $W_n = F_n$, and $V_n = L_n$.

3. We now consider a slight variant of the above. Put

$$E_{n,r} = \begin{vmatrix} F_r & F_{r+1} & F_{r+2} & \cdots & F_{r+n-1} \\ -F_{r+n-1} & F_r & F_{r+1} & \cdots & F_{r+n-2} \\ -F_{r+n-2} & -F_{r+n-1} & F_r & \cdots & F_{r+n-3} \\ \vdots & & & & \vdots \\ -F_{r+1} & -F_{r+2} & -F_{r+3} & \cdots & F_r \end{vmatrix}$$

We shall prove

$$E_{n,r} = \frac{(F_r + F_{n+r})^n + (-1)^n(F_{n+r-1} + F_{r-1})^n}{1 + L_n + (-1)^n} \tag{5}$$

A determinant of the form

$$S(a_0,\cdots,a_{n-1}) = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ -a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ -a_{n-2} & -a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & & & & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & a_0 \end{vmatrix}$$

is termed a skew circulant. Scott [1, Vol. 4, p. 356] has shown that

$$S(a_0,\cdots,a_{n-1}) = \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1} a_j\epsilon_k^j\right), \tag{6}$$

where the

$$\epsilon_k = \cos\frac{(2k+1)\pi}{2} + i\sin\frac{(2k+1)\pi}{2}$$

are the n^{th} roots of -1. To prove (6) quickly, multiply $S(a_0, \cdots, a_{n-1})$ by the Vandermonde determinant $|\epsilon_i^j|$ $(i,j = 0, 1, \cdots, n-1)$, and treat as in the proof of (3).

To evaluate $E_{n,r}$ let $a_j = F_{j+r}$. A development similar to Section 1 shows that

$$E_{n,r} = \prod_{k=0}^{n-1}\left(\sum_{j=0}^{n-1}\frac{\alpha^r(\alpha\epsilon_k)^j - \beta^r(\beta\epsilon_k)^j}{\alpha - \beta}\right)$$

(7)

$$= \prod_{k=0}^{n-1}\frac{F_{n+r} + F_r + [F_{n+r-1} + F_{r-1}]\epsilon_k}{(1 - \alpha\epsilon_k)(1 - \beta\epsilon_k)} .$$

For arbitrary x and y,

$$(8) \quad \prod_{k=0}^{n-1}(x - y\epsilon_k) = y^n\prod_{k=0}^{n-1}\left(\frac{x}{y} - \epsilon_k\right) = y^n\left[\left(\frac{x}{y}\right)^n + 1\right] = x^n + y^n .$$

Application of this to (7) yields the desired result (5).

We remark that as before (5) simplifies for even n. Ruggles [2] has shown that

$$F_{n+p} + F_{n-p} = \begin{cases} F_nL_p, & p \text{ even} \\ L_nF_p, & p \text{ odd} \end{cases} .$$

Then if $n \equiv 0 \pmod 4$,

$$E_{n,r} = \frac{F^n_{\frac{1}{2}n}\left(F^n_{r+\frac{1}{2}n} + F^n_{r-1+\frac{1}{2}n}\right)}{2 + L_n} .$$

while if $n \equiv 2 \pmod 4$,

$$E_{n,r} = \frac{F^n_{\frac{1}{2}n}\left(L^n_{r+\frac{1}{2}n} + L^n_{r-1+\frac{1}{2}n}\right)}{2 + L_n} .$$

Note that the latter yields on comparison with the determinant the identity

$$5(F^2_{r+1} + F^2_r) = L^2_{r+1} + L^2_r = 5F_{2r+1} .$$

4. The extension of this to second-order recurring sequences involves no new ideas, and the details are therefore omitted. Let W_n and V_n be as before, with the exception that we require a and b not be n^{th} roots of -1 rather than $+1$ to avoid division by zero. Put

$$E_{n,r}(W) = S(W_r, W_{r+1}, \cdots, W_{r+n-1}) .$$

Using (6) and (8), we find

$$E_{n,r}(W) = \frac{(W_{n+r} + W_r) + q^n(W_{n+r-1} + W_{r-1})^n}{1 + V_n + q^n} ,$$

which reduces to (5) when $q = -q = 1$, $W_n = F_n$, and $V_n = L_n$.

REFERENCES

1. Thomas Muir, The Theory of Determinants (4 Vols.), Dover, New York, 1960.
2. I. D. Ruggles, "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, 1 (1963), No. 2, pp. 75-80.
3. V. I. Smirnov, Linear Algebra and Group Theory, McGraw-Hill, New York, 1961.