FIBONACCI NUMBERS AND EULERIAN POLYNOMIALS

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Given a sequence of numbers $\{\lambda_0, \lambda_1, \cdots\}$ one can define a linear operator on Π, the set of all polynomials, by means of the symbolic relation

(1) $$\Lambda x^n = (x + \lambda)^n \qquad n = 0, 1, 2, \cdots ,$$

where it is understood that after expanding $(x + \lambda)^n$, we replace λ^k by λ_k. This operator can also be represented on Π by a differential operator of infinite order. Indeed, one can show that if

(2) $$\Lambda = \sum_{n=0}^{\infty} \frac{\lambda_n}{n!} D^n \qquad D = d/dx ,$$

then

$$\Lambda f(x) = f(x + \lambda) \qquad f \in \Pi .$$

A third representation of this operator can be obtained using a well-known theorem of Boas [2]. Given any sequence of numbers $\lambda_0, \lambda_1, \lambda_2, \cdots$ we can find a function of $\alpha(t)$ of bounded variation on $(0,\infty)$ such that

$$\lambda_n = \int_0^\infty t^n d\alpha(t) ,$$

so that

(3) $$\Lambda f(x) = \int_0^\infty f(x + t)d\alpha(t) \qquad f \in \Pi .$$

We shall refer to the sequence $\lambda_0, \lambda_1, \lambda_2, \cdots$ as the sequence of moments corresponding to the operator Λ.

Naturally, all the representations (1), (2), and (3) are valid (and define the same operator) on Π. However, we can extend the definition formally to functions with power series expansion.

In this note, we are interested in a class of "mean operators" defined by

$$(4) \qquad Mf(x) = \mu f(x + c_1) + (1 - \mu)f(x + c_2) \ ,$$

where μ, c_1, c_2 are given numbers. Obviously, M takes polynomials into polynomials of the same degree.

To determine the corresponding moments, we note that

$$Mx^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}[\mu c_1^k + (1 - \mu)c_2^k] = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} m_k \ ,$$

so that

$$m_n = \mu c_1^n + (1 - \mu)c_2^n \qquad n = 0, 1, 2, \cdots \ .$$

It is easy to verify that

$$(5) \qquad \begin{aligned} m_{n+1} &= (c_1 + c_2)m_n - c_1 c_2 m_{n-1} \qquad n = 1, 2, \cdots \\ m_0 &= 1, \quad m_1 = \mu c_1 + (1 - \mu)c_2 \ . \end{aligned}$$

To find the inverse operator M^{-1}, put

$$(6) \qquad M^{-1} = \sum_{k=0}^{\infty} \frac{m_k^*}{k!} D^k \ .$$

Then $M^{-1}Mx^n = x^n$ for all n imply that

(7) $$\sum_{k=0}^{j} \binom{j}{k} m_k m^*_{j-k} = 0 \text{ (if } j > 0) \quad \text{and} \quad 1 \text{ (if } j = 0) .$$

Thus, if we multiply (7) by $t^n/n!$ and sum over all integers $n \geq 0$, we get

$$\left[\sum_{k=0}^{\infty} \frac{m_k}{k!} t^k\right]\left[\sum_{k=0}^{\infty} \frac{m^*_k}{k!} t^k\right] = 1 ,$$

so that

$$\sum_{k=0}^{\infty} \frac{m^*_k}{k!} t^k = \frac{1}{\mu e^{c_1 t} + (1 - \mu)e^{c_2 t}} .$$

If we recall the Eulerian polynomials [1] defined by means of

$$\frac{1 - \lambda}{e^t - \lambda} e^{xt} = \sum_{n=0}^{\infty} H_n(x|\lambda)t^n/n! ,$$

then we see that

$$m^*_n = (c_2 - c_1)^n H_n\left(\frac{c_1}{c_1 - c_2} \,\middle|\, \frac{\mu}{\mu - 1}\right) .$$

Thus the operator inverse to the operator (1) is given by

(8) $$M^{-1} = \sum_{n=0}^{\infty} \frac{(c_2 - c_1)^n}{n!} H_n\left(\frac{c_1}{c_1 - c_2} \,\middle|\, \frac{\mu}{\mu - 1}\right) D^n .$$

In particular, if we take $\mu = 1/2$, $c_1 = (1 + \sqrt{5})/2$, $c_2 = (1 - \sqrt{5})/2$ in the above, we see that $m_n = F_{n+1}$. Thus the Fibonacci numbers F_{n+1},

$n = 0, 1, 2, \cdots$ are the moments corresponding to the mean operator

$$\delta f(x) = \frac{1}{2} \left\{ f\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right) + f\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) \right\} \tag{9}$$

$$= \sum_{n=0}^{\infty} \frac{F_{n+1}}{n!} D^n f(x) \quad .$$

If we note that $H_n(x|-1) = E_n(x)$, the Euler polynomials generated by

$$\frac{2e^{xt}}{e^t + 1} ,$$

we find that the operator inverse to δ is

$$\delta^{-1} f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{5^{n/2}}{n!} E_n\left(\frac{1 + \sqrt{5}}{2\sqrt{5}}\right) D^n f(x) \quad .$$

The moments corresponding to δ^{-1} are the numbers

$$(-1)^n 5^{n/2} E_n\left(\frac{1 + \sqrt{5}}{2\sqrt{5}}\right) \qquad n = 0, 1, 2, \cdots \quad .$$

Another special case is when $\mu = 1/2$, $c_1 = 0$, $c_2 = 1$. We get that $\lambda_0 = 1$, $\lambda_k = 1/2$ $(k > 0)$ are the moments of the operator

$$\delta f(x) = \frac{1}{2} \{f(x) + f(x + 1)\} \quad . \tag{10}$$

The moments of δ^{-1} have, therefore, the generating relation

$$\frac{2}{1 + e^t} = \sum_{n=0}^{\infty} \lambda_n^* t^n/n! \quad ,$$

and thus

$$\lambda_0^* = 1, \qquad \lambda_n^* = (1 - 2^n)\frac{B_n}{n} \qquad (n \geq 1),$$

where B_n are the Bernoulli numbers.

Similarly, if we consider another mean operator, namely,

$$Lf(x) = \frac{1}{n}\sum_{k=0}^{n-1} f(x + kh), \tag{11}$$

we see that the moments corresponding to L are the numbers

$$\ell_0 = 1, \qquad \ell_m = \frac{h^m}{n}\sum_{k=0}^{n-1} k^m = \frac{h^m}{n}\left\{\frac{B_{m+1}(n) - B_{m+1}}{m+1}\right\},$$

where $B_m(x)$ are the Bernoulli polynomials and $B_m = B_m(0)$.

The moments corresponding to the inverse operator L^{-1} are

$$\ell_m^* = h^m \sum_{k=0}^{m}\binom{m}{k}\frac{B_k}{m-k+1}\, n^k \qquad (m = 0, 1, 2, \cdots).$$

REFERENCES

1. L. Carlitz, "Eulerian Numbers and Polynomials," Mathematics Magazine, Vol. 30 (1959), pp. 247-260.
2. R. P. Boas, "The Stieltjes Moment Problem for Functions of Bounded Variation," Bulletin of the American Mathematical Society, Vol. 45 (1939), pp. 399-404.