SOME SUMMATION FORMULAS*

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1. Multiple summation formulas of a rather unusual kind can be obtained in the following way. Let

$$f(x) = 1 - a_1x - a_2x^2 - \cdots \tag{1.1}$$

denote a series that converges for small x. Put

$$\frac{1}{f(x)} = 1 + b_1x + b_2x^2 + \cdots , \tag{1.2}$$

so that

$$\frac{1}{f(x)(1-y)} = \sum_{m,n=0}^{\infty} b_m x^m y^n \qquad (b_0 = 1) . \tag{1.3}$$

Replacing y by $x^{-1}y$, Eq. (1.3) becomes

$$\frac{1}{f(x)(1 - x^{-1}y)} = \sum_{m,n=0}^{\infty} b_m x^{m-n} y^n . \tag{1.4}$$

Let k denote a fixed non-negative integer. Then that part of the right-hand side of (1.4) that contains terms in x^{-k} is evidently

$$\sum_{m=0}^{\infty} b_m y^{m+k} = y^k/f(y) . \tag{1.5}$$

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On the other hand, since

$$(1 - a_1x - a_2x^2 - \cdots)(1 - x^{-1}y) = (1 + a_1y) - x^{-1}y - (a_1 - a_2y)x - (a_2 - a_3y)x^2 - \cdots$$

It follows that

$$\frac{1}{f(x)(1 - x^{-1}y)} = \sum_{n=0}^{\infty} \frac{[x^{-1}y + (a_1 - a_2y)x + (a_2 - a_3y)x^2 + \cdots]^{n+1}}{(1 + a_1y)^{n+1}}$$

$$= \sum_{r=0}^{\infty} \sum_{s_j=0}^{\infty} \frac{(r + s_1 + s_2 + \cdots)!}{r!\, s_1!\, s_2! \cdots} \cdot \frac{y^r (a_1 - a_2y)^{s_1} (a_2 - a_3y)^{s_2} \cdots}{(1 + a_1y)^{r+s_1+s_2+\cdots+1}} \cdot x^{-r+s_1+2s_2+3s_3+\cdots} .$$

The part of the multiple summation on the right that contains terms in x^{-k} is obtained by taking

$$r = k + s_1 + 2s_2 + 3s_3 + \cdots .$$

Comparison with (1.5) therefore yields the following identity:

$$\sum_{s_j=0}^{\infty} \frac{(k + 2s_1 + 3s_2 + 4s_3 + \cdots)!}{s_1!\, s_2! \cdots (k + s_1 + 2s_2 + 3s_3 + \cdots)!} \cdot \frac{y^{s_1+2s_2+3s_3+\cdots} (a_1 - a_2y)^{s_1} (a_2 - a_3y)^{s_2}}{(1 + a_1y)^{2s_1+3s_2+\cdots}} \tag{1.6}$$

$$= \frac{(1 + a_1y)^{k+1}}{1 - a_1y - a_2y^2 - \cdots} .$$

If we take

$$z_j = a_j y^j \qquad (j = 1, 2, 3, \cdots) ,$$

(1.6) becomes

$$(1.7)\quad \sum_{s_j=0}^{\infty} \frac{(k+2s_1+3s_2+4s_3+\cdots)!}{s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!} \cdot \frac{(z_1-z_2)^{s_1}(z_2-z_3)^{s_2}\cdots}{(1+z_1)^{2s_1+3s_2+\cdots}} = \frac{(1+z)^{k+1}}{1-z_1-z_2-\cdots}.$$

If we now put

$$z_j - z_{j+1} = u_j \qquad (j = 1, 2, 3, \cdots) ,$$

so that

$$z_j = u_j + u_{j+1} + u_{j+2} + \cdots \qquad (j = 1, 2, 3, \cdots) ,$$

we get

$$(1.7)\quad \sum_{s_j=0}^{\infty} \frac{(k+2s_1+3s_2+4s_3+\cdots)!}{s_1!s_2!\cdots(k+s_1+2s_2+\cdots)!} \cdot \frac{u_1^{s_1}u_2^{s_2}\cdots}{(1+u_1+u_2+u_3+\cdots)^{2s_1+3s_2+\cdots}}$$

$$= \frac{(1+u_1+u_2+\cdots)^{k+1}}{1-u_1-2u_2-3u_3-\cdots} ,$$

where

$$u_1 + u_2 + u_3 + \cdots$$

is absolutely convergent.

2. There are numerous special cases of the above identities that may be noted. To begin with, we take

$$u_3 = u_4 = \cdots = 0 .$$

Changing the notation slightly, Eq. (1.7) gives

(2.1) $$\sum_{r,s=0}^{\infty} \frac{(k + 2r + 3s)!}{r!\, s!\, (k + r + 2s)!} \cdot \frac{u^r v^s}{(1 + u + v)^{2r+3s}} = \frac{(1 + u + v)^{k+1}}{1 - u - 2v} .$$

In particular, for $v = 0$, Eq. (2.1) reduces to

(2.2) $$\sum_{r=0}^{\infty} \frac{(k + 2r)!}{r!\, (k + r)!} \frac{u^r}{(1 + u)^{2r}} = \frac{(1 + u)^{k+1}}{1 - u} .$$

This is easily verified for $k = 0$. Indeed,

$$\sum_{r=0}^{\infty} \binom{2r}{r} \frac{u^r}{(1 + u)^{2r}} = \left\{1 - \frac{4u}{(1 + u)^2}\right\}^{-\frac{1}{2}} = \frac{1 + u}{1 - u} ,$$

in agreement with the special case of (2.2).

If we take all $u_j = 0$ except u_{p-1}, we get

(2.3) $$\sum_{s=0}^{\infty} \binom{k + ps}{s} \frac{u^s}{(1 + u)^{ps}} = \frac{(1 + u)^{k+1}}{1 - (p - 1)u} .$$

Summations like (2.3) are usually obtained by means of the Lagrange-Burmann expansion formula. For example, it is proved [1, p. 126, No. 216] that

(2.4) $$\sum_{n=0}^{\infty} \binom{\alpha + \beta n}{n} w^n = \frac{(1 + z)^{\alpha}}{1 - \beta w(1 + z)^{\beta+1}} ,$$

where

(2.5) $$w = \frac{z}{(1 + z)^{\beta}} .$$

Making use of (2.5), the right member of (2.4) is seen to be equal to

$$\frac{(1 + z)^{\alpha+1}}{1 - (\beta - 1)z} ,$$

so that (2.4) is in agreement with (2.3).

It should be observed that (2.3) has been proved above only for integral $k \geq 0$, $p \geq 1$. However, since

$$\sum_{s=0}^{\infty} \binom{k + ps}{s} \frac{u^s}{(1 + u)^{k+ps+1}} = \sum_{s=0}^{\infty} \binom{k + ps}{s} u^s \sum_{r=0}^{\infty} (-1)^r \binom{k + r + ps}{r} u^r$$

$$= \sum_{n=0}^{\infty} u^n \sum_{s=0}^{n} (-1)^{n-s} \binom{k + ps}{s} \binom{k + n - s + ps}{n - s} ,$$

it follows that (2.3) is equivalent to

$$\sum_{s=0}^{n} (-1)^{n-s} \binom{k + ps}{s} \binom{k + n - s + ps}{n - s} = (p - 1)^n . \tag{2.6}$$

Since (2.6) is a polynomial identity that holds for

$$k = 0, 1, 2, \cdots; \qquad p = 1, 2, 3, \cdots ,$$

it therefore holds for arbitrary k, p.

3. The proof that (2.3) holds for arbitrary k, p suggests that (1.7) also holds for arbitrary k. We divide both sides of (1.7) by

$$(1 + u + u_2 + \cdots)^{k+1} .$$

Then since

$$(1 + u_1 + u_2 + \cdots)^{-k-2s_1-3s_2-\cdots-1}$$

$$= \sum_{r=0}^{\infty} (-1)^r \binom{k + r + 2s_1 + 3s_2 + \cdots}{r} (u_1 + u_2 + \cdots)^r$$

$$= \sum_{r_j=0}^{\infty} \binom{k + r + 2s_1 + 3s_2 + \cdots}{r} \frac{r!}{r_1!\, r_2! \cdots} u_1^{r_1} u_2^{s_2} \cdots ,$$

where $r = r_1 + r_2 + \cdots$, it follows that the left member of (1.7) is equal to

$$\sum_{r_j=0}^{\infty} (-1)^r \sum_{s_j=0}^{\infty} \frac{(k + 2s_1 + 3s_2 + \cdots)!}{s_1!\, s_2! \cdots (k + s_1 + 2s_2 + \cdots)!} \binom{k + r + 2s_1 + 3s_2 + \cdots}{r} \frac{r!}{r_1!\, r_2!} \cdot u_1^{r_1+s_1} u_2^{r_2+s_2+\cdots} .$$

Hence (1.7) is equivalent to

$$\sum_{r_j+s_j=n_j}^{\infty} (-1)^r \binom{k + 2s_1 + 3s_2 + \cdots}{s} \binom{k + r + 2s_1 + 3s_2 + \cdots}{r} \cdot \frac{s!}{s_1!\, s_2! \cdots} \cdot \frac{r!}{r_1!\, r_2! \cdots}$$

(3.1)

$$= \frac{(n_1 + n_2 + \cdots)!}{n_1!\, n_2! \cdots} 1^{n_1} 2^{n_2} 3^{n_3} ,$$

where

$$r = r_1 + r_2 + \cdots , \qquad s = s_1 + s_2 + \cdots .$$

Since (3.1) is a polynomial identity in k, it is valid for arbitrary k. Therefore (1.7) is proved for arbitrary k.

4. Another special case of (1.7) that is of some interest is obtained by taking all $u_j = 0$ except u_{p-1} and u_{q-1}. We evidently get

$$(4.1)\quad \sum_{r,s=0}^{\infty} \binom{k+pr+qs}{r+s}\binom{r+s}{r} \frac{u^r v^s}{(1+u+v)^{pr+qs}} = \frac{(1+u+v)^{k+1}}{1-(p-1)u-(q-1)v} \quad (q \neq p).$$

As above, we can assert that (4.1) holds for all k, p, q. This can evidently be extended in an obvious way, thus furnishing extensions of (2.3) involving an arbitrary number of parameters.

We remark that (4.1) is equivalent to

$$(4.2)\quad \sum_{\substack{r+i=m \\ s+j=n}} (-1)^{i+j} \binom{k+pr+qs}{r+s}\binom{k+pr+qs+i+j}{i+j}\binom{r+s}{r}\binom{i+j}{i} = \binom{m+n}{m}(p-1)^m (q-1)^n ,$$

which is itself a special case of (3.1).

REFERENCE

1. G. Pólya and G. Szegö, Aufaben und Lehrsätze aus der Analysis, 1, Berlin, 1925.

LETTER TO THE EDITOR

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In the note by W. R. Spickerman, "A Note on Fibonacci Functions," Fibonacci Quarterly, October, 1970, pp. 397-401, his Theorem 1, p. 397, states that if $f(x)$ is a Fibonacci function, i.e.,

$$(1)\quad f(x+2) = f(x+1) + f(x),$$

then $\int f(x)dx$ is also a Fibonacci function. Since $\int f(x)dx = h(x) + C$, where C is the arbitrary constant of integration, the above result assumes that $C = 0$. Thus, a formulation of this result in terms of a definite integral seems apropos.

[Continued on page 40.]