ON ITERATIVE FIBONACCI SUBSCRIPTS

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The Fibonacci sequence is defined by the recurrence relation $F_n + F_{n+1} = F_{n+2}$ and the initial values $F_1 = F_2 = 1$.

The main result of this paper is

Theorem 4. For positive integers a, k, m and n such that $k \geq m$,

$$\left(\underbrace{F_{aF_{a_{\cdot_{\cdot_{\cdot_{F_a}}}}}}}_{n}\right)^k \text{ divides } \underbrace{F_{aF_{a_{\cdot_{\cdot_{\cdot_{F_a}}}}}}}_{mn} \cdot \left(\underbrace{F_{aF_{a_{\cdot_{\cdot_{\cdot_{F_a}}}}}}}_{n}\right)^{k-m} .$$

The proof of Theorem 4 will depend on all results preceding it in this paper.

Let N be the set of natural numbers.

Definition 1. For any a, b in N the symbol $f_n(a,b)$ is defined for each n in N as follows:

i) $$f_1(a,b) = F_{ab}$$

ii) $$f_{n+1}(a,b) = f_1(a, f_n(a,b)) .$$

By induction, we observe that

$$f_n(a,b) = \underbrace{F_{aF_{a_{\cdot_{\cdot_{\cdot_{F_{aF_{ab}}}}}}}}}_{n} .$$

Definition 2. For any a in N the symbol $f_n(a)$ is defined for each n in N as follows:

i) $$f_1(a) = F_a$$

ii) $$f_{n+1}(a) = f_1(a, f_n(a)) .$$

By induction, we observe that $f_n(a) = f_n(a,1)$.

If a,b are in N, we write $a|b$ if and only if there exists some c in N such that $b = ac$.

In the sequel we shall let a,b and c denote arbitrary elements of N.

Lemma 1. If $b|c$, then $f_1(a,b)|f_1(a,c)$ for all a in N.

Proof. If $b|c$, then $ab|ac$ for all a in N. From Hardy and Wright [1, p. 148] we have, if $n > 0$, then $F_n|F_{rn}$ for every $r > 0$. So in the present notation $f_1(a,b) = F_{ab}|F_{ac} = f_1(a,c)$ for all a in N.

Lemma 2. If $b|f_1(a,c)$, then $bf_1(a,c)|f_1(a,bc)$.

Proof. From Vinson [2] we have in the present notation,

$$F_{acb} = \sum_{j=1}^{b} \binom{b}{j} F_{ac}^{j} F_{ac-1}^{b-j} F_j \;.$$

For $j = 1$, we have

$$bF_{ac} \,\Big|\, \binom{b}{1} F_{ac} F_{ac-1}^{b-1} F_1 \;.$$

For $j > 1$, we have, since $b|f_1(a,c) = F_{ac}$, that

$$bF_{ac} \,\Big|\, F_{ac}^2 \,\Big|\, \sum_{j=2}^{b} \binom{b}{j} F_{ac}^{j} F_{ac-1}^{b-j} F_j \;.$$

Thus $bf_1(a,c) = bF_{ac}|F_{acb} = f_1(a,bc)$.

Corollary 1. If $b|f_{n+1}(a)$, then $bf_{n+1}(a)|f_1(a, bf_n(a))$.

Corollary 2. If $b|f_1(a)$, then $bf_1(a)|f_1(a,b)$.

Theorem 1. If $b|c$, then $f_n(a,b)|f_n(a,c)$.

Proof. We use induction on n. The case $n = 1$ is true by Lemma 1, Suppose $f_q(a,b)|f_q(a,c)$. Then by Lemma 1 and Definition 1,

$$f_1(a, f_q(a,b)) = f_{q+1}(a,b)|f_{q+1}(a,c) = f_1(a, f_q(a,c)) \;.$$

Corollary 3. $f_n(a) | f_n(a,c)$.

Theorem 2. $f_m(a, f_n(a,b)) = f_{m+n}(a,b)$.

Proof. We use induction on m. The case $m = 1$ is true by Definition 1. Suppose $f_q(a, f_n(a,b)) = f_{q+n}(a,b)$. Then by Definition 1,

$$f_{q+1}(a, f_n(a,b)) = f_1(a, f_q(a, f_n(a,b))) = f_1(a, f_{q+n}(a,b)) = f_{q+1+n}(a,b) .$$

Corollary 4: $f_m(a, f_n(a)) = f_{m+n}(a)$.

Lemma 3. $f_n(a) | f_{m+n}(a)$ for $m \geq 0$.

Proof. The case $m = 0$ is clear. Suppose $m > 0$. Then by corollaries 3 and 4,

$$f_n(a) | f_n(a, f_m(a)) = f_{m+n}(a) .$$

Lemma 4. $f_n(a) f_n(a) | f_{2n}(a)$.

Proof. We use induction on n. By corollary 2 and definition 2, $f_1(a) | f_1(a)$ implies

$$f_1(a) f_1(a) | f_1(a, f_1(a)) = f_2(a) ,$$

so the case $n = 1$ is true. Suppose $f_q(a) f_q(a) | f_{2q}(a)$. Then by Lemma 1,

$$f_1(a, f_q(a) f_q(a)) | f_1(a, f_{2q}(a)) = f_{2q+1}(a)$$

and by Lemma 1 again,

$$f_1(a, f_1(a, f_q(a) f_q(a))) | f_1(a, f_{2q+1}(a)) = f_{2(q+1)}(a) . \tag{1}$$

Since $f_{q+1}(a) | f_{q+1}(a)$ we have, by Corollary 1,

$$f_{q+1}(a) f_{q+1}(a) | f_1(a, f_{q+1}(a) f_q(a)) . \tag{2}$$

By Lemma 3, $f_q(a) | f_{q+1}(a)$ so by Corollary 1, $f_q(a) f_{q+1}(a) | f_1(a, f_q(a) f_q(a))$. Therefore, by Lemma 1,

$$f_1(a, f_q(a)f_{q+1}(a)) \mid f_1(a, f_1(a, f_q(a)f_q(a))) .$$

By Equations (1) and (2), the proof is complete.

Theorem 3. $f_m(a)f_n(a) \mid f_{m+n}(a)$.

Proof. It is sufficient to prove the theorem for all $n \geq m$. Let $n = m + r$ where $r \geq 0$. We use induction on r. The case $r = 0$ is true by Lemma 4. Suppose $f_m(a)f_{m+q}(a) \mid f_{2m+q}(a)$ for $q \geq 0$. Then, by Lemma 1,

$$(3) \qquad f_1(a, f_m(a)f_{m+q}(a)) \mid f_1(a, f_{2m+q}(a)) = f_{2m+q+1}(a) .$$

By Lemma 3, $f_m(a) \mid f_{m+q+1}(a)$, so by Corollary 1,

$$f_m(a)f_{m+q+1}(a) \mid f_1(a, f_m(a)f_{m+q}(a)) .$$

By Equation (3), the proof is complete.

Lemma 5. $f_{m+n}(a) \mid f_m(a, f_n^k(a))$ for $k > 0$.

Proof. By Theorem 1, and Corollary 4, $f_n(a) \mid f_n^k(a)$ implies

$$f_m(a, f_n(a)) = f_{m+n}(a) \mid f_m(a, f_n^k(a)) .$$

Lemma 6. $f_n(a)f_m(a, f_n^k(a)) \mid f_m(a, f_n^{k+1}(a))$ for $k \geq 0$.

Proof. The case $k = 0$ is true by Theorem 3 and Corollary 4. Suppose $k > 0$. We now use induction on m. By Lemmas 3 and 5,

$$f_n(a) \mid f_{n+1}(a)\ f_1(a, f_n^k(a))$$

for $k > 0$. So by Lemma 2,

$$f_n(a)f_1(a, f_n^k(a)) \mid f_1(a, f_n(a)f_n^k(a)) = f_1(a, f_n^{k+1}(a)) .$$

So the case $m = 1$ is true. Suppose

$$f_n(a)f_q(a, f_n^k(a)) \mid f_q(a, f_n^{k+1}(a))$$

for $k > 0$. Then by Lemma 1,

$$f_1(a, f_n(a)f_q(a, f_n^k(a))) \mid f_1(a, f_q(a, f_n^{k+1}(a))) = f_{q+1}(a, f_n^{k+1}(a)) \tag{4}$$

by Definition 1. By Lemmas 3 and 5,

$$f_n(a) \mid f_{q+1+n}(a) \mid f_{q+1}(a, f_n^k(a)) = f_1(a, f_q(a, f_n^k(a)))$$

for $k > 0$, which implies by Lemma 2 that

$$f_n(a)f_{q+1}(a, f_n^k(a)) \mid f_1(a, f_n(a)f_q(a, f_n^k(a))) .$$

So by Eq. (4), the proof is complete.

Lemma 7. $f_n^k(a) \mid f_n(a, f_n^{k-1}(a))$ for $k > 0$.

Proof. We use induction on k. The case $k = 1$ is clear. Suppose

$$f_n^q(a) \mid f_n(a, f_n^{q-1}(a))$$

for $q > 0$. Then

$$f_n^{q+1}(a) \mid f_n(a)f_n(a, f_n^{q-1}(a)) \mid f_n(a, f_n^q(a))$$

for $q - 1 \geq 0$, by Lemma 6.

Theorem 4. $f_n^k(a) \mid f_{mn}(a, f_n^{k-m}(a))$ for $k \geq m > 0$.

Proof. We use induction on m. The case $m = 1$ is true by Lemma 7. Suppose

$$f_n^k(a) \mid f_{qn}(a, f_n^{k-q}(a))$$

for $k \geq q > 0$. Then by Theorems 1 and 2,

$$f_n(a, f_n^k(a)) \mid f_n(a, f_{qn}(a, f_n^{k-q}(a))) = f_{(q+1)n}(a, f_n^{k+1-(q+1)}(a)), \tag{5}$$

where $k + 1 \geq q + 1 > 0$. By Lemma 7,

$$f_n^{k+1}(a) \mid f_n(a, f_n^k(a))$$

for $k + 1 > 0$. Therefore, by Eq. (5),

$$f_n^{k+1}(a) \mid f_{(q+1)n}(a, f_n^{k+1-(q+1)}(a))$$

for $k + 1 \geq q + 1 > 0$, and the proof is complete.

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REFERENCES

1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, London, 1954.
2. John Vinson, "The Relation of the Period Modulo m to the Rank of Apparition of m in the Fibonacci Sequence," Fibonacci Quarterly, Vol. 1, No. 2, April 1963, p. 38.

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Theorem. Let f(x) be a Fibonacci function (see [1]). Then,

(2) $$\int_1^2 f(t)dt = A \qquad (A \text{ is a constant}),$$

is a necessary and sufficient condition that

(3) $$g(x) = \int_0^x f(t)dt + A, \qquad g(0) = A,$$

also be a Fibonacci function.

Proof. Necessity. If $g(x)$ is a Fibonacci function, then $g(x + 2) = g(x + 1) + g(x)$. For $x = 0$, $g(2) = g(1) + g(0)$, which simplifies to (2).

Sufficiency. By integration, we have

$$\int_0^x f(t + 2)dt = \int_0^x f(t + 1)dt + \int_0^x f(t)dt .$$

Let $t + 2 = u$ and $t + 1 = v$ to obtain

(4) $$\int_2^{x+2} f(u)du = \int_1^{x+1} f(v)dv + \int_0^x f(t)dt .$$

Using (3), we obtain from (4), $g(x + 2) = g(x + 1) + g(x)$, by using (2).