A SYMMETRIC SUBSTITUTE FOR STIRLING NUMBERS
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Let

\[ \binom{x}{0} = 1 \]

and

\[ \binom{x}{r} = \frac{x(x-1)(x-2) \cdots (x-r+1)}{1 \cdot 2 \cdot 3 \cdots r} \]

for all complex numbers \( x \) and all positive integers \( r \). It is well known that

\[ (0) + \frac{1}{1!} + \frac{2}{2!} + \cdots + \frac{n}{n!} = \frac{(n+1)}{r+1}, \tag{1} \]

and that for every non-negative integer \( d \) there exist integers \( s_{d0}, s_{d1}, \ldots, s_{dd} \) such that

\[ x^d = s_{d0} \binom{x}{0} + s_{d1} \binom{x}{1} + s_{d2} \binom{x}{2} + \cdots + s_{dd} \binom{x}{d} \tag{2} \]

holds for all \( x \). (The \( s_{dj} \) are related to the Stirling numbers of the second kind.) Using (1) and (2), one obtains the summation formulas

\[ 0^d + 1^d + 2^d + \cdots + n^d = s_{d0} \binom{n+1}{1} + s_{d1} \binom{n+1}{2} + \cdots + s_{dd} \binom{n+1}{d+1}. \tag{3} \]

This paper presents alternates for (2) and (3) in which the \( s_{dj} \) are replaced by coefficients having symmetry properties and other advantages. Part of the work generalizes with the help of Dov Jarden’s results from the \( \binom{n}{r} \) to generalized binomial coefficients.
Using the well known

\[
\binom{x}{r} + \binom{x}{r+1} = \binom{x+1}{r+1},
\]

one easily proves

\[
(4) \left( x \atop r-s \right) = \binom{s}{0} \binom{s-1}{r-1} + \cdots + (-1)^s \binom{s}{r}
\]

by mathematical induction. Then (2) and (4) imply that for every non-negative integer \( d \) there exist integers \( a_{dj} \) such that

\[
(5) \quad x^d = s_{d0} \binom{x}{d} + a_{d1} \binom{x+1}{d} + \cdots + a_{dd} \binom{x+d}{d}.
\]

From (1) and (5), one now obtains

\[
(6) \quad 0^d + 1^d + \cdots + n^d = a_{d0} \binom{x+1}{d+1} + a_{d1} \binom{x+2}{d+1} + \cdots + a_{dd} \binom{x+d+1}{d+1}.
\]

For example,

\[
x^2 = \binom{x}{2} + \binom{x+1}{2}, \quad x^3 = \binom{x}{3} + 4 \binom{x+1}{3} + \binom{x+2}{3},
\]

\[
x^4 = \binom{x}{4} + 11 \binom{x+1}{4} + 11 \binom{x+2}{4} + \binom{x+3}{4},
\]

\[
x^5 = \binom{x}{5} + 26 \binom{x+1}{5} + 66 \binom{x+2}{5} + 26 \binom{x+3}{5} + \binom{x+4}{5},
\]

\[
0^2 + 1^2 + 2^2 + \cdots + n^2 = \binom{n+1}{3} + \binom{n+2}{3}.
\]

and

\[
\sum_{k=0}^{n} k^3 = \binom{n+1}{4} + 4 \binom{n+2}{4} + \binom{n+3}{4}.
\]
The listed cases of (5) suggest that the following may be true:

\[ a_{dd} = 0 \]

\[ a_{d0} = 1 = a_{d,d-1} \]

\[ a_{dj} = a_{d,d-1-j} \]

\[ a_{d0} + a_{d1} + \cdots + a_{d,d-1} = 1 \cdot 2 \cdot 3 \cdots d = d! \]

Successively letting \( x \) be 0, -1, 1, -2, 2, \( \cdots \) in (5) establishes (7), (8), and (with the help of mathematical induction) the symmetry formula (9). These substitutions also prove that the \( a_{dj} \) are unique. One obtains (10) from

\[
\frac{1}{d+1} = \int_0^1 x^d \, dx = \lim_{n \to \infty} \left( \left( \frac{1}{n} \right)^d + \left( \frac{2}{n} \right)^d + \cdots + \left( \frac{n}{n} \right)^d \right) / n
\]

\[
= \lim_{n \to \infty} \left[ \frac{(1^d + 2^d + \cdots + n^d)}{n^{d+1}} \right]
\]

\[
= \frac{a_{d0} + a_{d1} + \cdots + a_{d,d-1}}{(d+1)!}
\]

A recursion formula for the \( a_{dj} \) is derived as follows:

\[
\sum_{j=0}^{d} a_{d+1,j} \binom{x+j}{d+1} = x^{d+1}
\]

\[
= x \sum_{j=0}^{d-1} a_{dj} \binom{x+j}{d}
\]

\[
= \sum_{j=0}^{d-1} a_{dj} [x - d + j + (d - j)] \binom{x+j}{d}
\]
This and uniqueness of the $a_{dj}$ imply that for $j = 1, 2, \cdots, d - 1$ one has

\[(11) \quad a_{d+1,j} = (j + 1)a_{d,j} + (d - j + 1)a_{d,j-1}.\]

Using $a_{d0} = 1$ and (11) gives us $a_{d+1,1} = 2a_{d1} + d$. Let $E$ be the operator on functions of $d$ such that $Ey_d = y_{d+1}$. Then $(E - 2)a_{d1} = d$ and

\[(E^2 - 2E + 1)(E - 2)a_{d1} = (d + 2) - 2(d + 1) + d = 0.\]

It follows from the theory of linear homogeneous difference equations with constant coefficients that there are constants $e_0$, $e_1$, and $e_2$ such that

\[a_{d1} = e_0 + e_1d + e_2 \cdot 2^d \quad \text{for} \quad d = 1, 2, 3, \cdots.\]
Using the known values of \(a_{11}, a_{21}\), and \(a_{31}\) one solves for \(e_0, e_1,\) and \(e_2\) and thus shows that \(a_{d1} = 2^d - d - 1\). Similarly, one sees that

\[(E - 1)^2(E - 2)^2(E - 3)a_{d2} = 0\]

and hence that there are constants \(f_1\) such that

\[a_{d2} = (f_0 + f_4d + f_5d^2) + (f_3 + f_7d)2^d + f_5 \cdot 3^d.\]

Determining the \(f_1\), one finds that

\[a_{d2} = 3^d - \binom{d + 1}{1} 2^d + \binom{d + 1}{2}.\]

Now (or after additional cases) one conjectures that

\[(12) \quad a_{dj} = \sum_{k=0}^{j} (-1)^k(j + 1 - k)^d \binom{d + 1}{k}.\]

Because of the symmetry formula (9), we know that (12) is equivalent to

\[(13) \quad a_{dj} = \sum_{k=0}^{d-j-1} (-1)^k(d - j - k)^d \binom{d + 1}{k}.\]

Substituting (13) into (5) gives us

\[(14) \quad x^d = \sum_{j=0}^{d-1} \left\{ \left[ \sum_{k=0}^{d-j-1} (-1)^k(d - j - k)^d \binom{d + 1}{k} \right] \left( x - j \right) \right\}.\]

Since the \(a_{dj}\) that satisfy (5) are unique, one can prove (13) and (12) by showing that (14) is an identity in \(x\). Since both sides of (14) are polynomials in
x of degree d, it suffices to verify (14) for the d + 1 values \( x = 0, 1, \cdots, d \). For such an x, (14) becomes

\[
x^d = x^d + \sum_{r=1}^{x-1} \left( \sum_{j=0}^{x-r} (-1)^j \binom{d+1}{j} \binom{x+d-r-j}{d} \right).
\]

Since \( \binom{a}{d} = 0 \) for \( a = 0, 1, \cdots, d-1 \), one has

\[
\sum_{j=0}^{x-r} (-1)^j \binom{d+1}{j} \binom{x+d-r-j}{d} = \sum_{j=0}^{d+1} (-1)^j \binom{d+1}{j} \binom{x+d-r-j}{d}.
\]

The right side sum in (16) is zero since it is a \((d + 1)^{st}\) difference of a polynomial of degree d. Hence (15) becomes the tautology \( x^d = x^d \). This establishes (13) and (12).

We next apply some of the above material to convolution formulas. It is well known (and easily shown by Maclaurin's expansion or Newton's binomial expansion) that

\[
(1 - x)^{-d-1} = \binom{d}{d} + \binom{d+1}{d} x + \binom{d+2}{d} x^2 + \cdots \quad \text{for } -1 < x < 1.
\]

Using (5) and (17), we obtain

\[
(a_d d - 1 + a_{d,d-2} x + \cdots + a_{d,0} x^{d-1})(1-x)^{-d-1} = 1 + 2^d x + 3^d x^2 + \cdots,
\]

where |x| < 1.

Now let

\[
p(d, x) = a_{d,0} + a_{d,1} x + \cdots + a_{d,d-1} x^{d-1} - a_{d,1} + a_{d,d-1} x + \cdots + a_{d,0} x^{d-1}.
\]

Then (18) can be rewritten as
\[ p(d, x) (1 - x)^{-d-1} = \sum_{j=0}^{\infty} (j + 1)^{d} x^{j}, \quad |x| < 1. \]

Also let

\[ p(d, x)p(e, x) = q(d, e, x) = c_{d,e,0} + c_{d,e,1} x + \ldots + c_{d,e,d+e-2} x^{d+e-2}. \]

Then

\[ \sum_{k=0}^{n} k^{d}(n - k)^{e} \]

is the coefficient of \( x^n \) in the Maclaurin expansion of

\[ q(d, e, x)(1 - x)^{-d-e-2}, \]

i.e., (22) is equal to

\[ \sum_{j=0}^{d+e-2} c_{d,e,j} \left( \frac{n + 1 + j}{d + e + 1} \right). \]

For example, since \( p(3, x) = 1 + 4x + x^2 \), and \( p(2, x) = 1 + x \), we have

\[ q(3, 2, x) = (1 + 4x + x^2)(1 + x) = 1 + 5x + 5x^2 + x^3, \]

and it follows from the equality of (22) and (23) that

\[ \sum_{k=0}^{n} k^{3}(n - k)^{2} = \left( \frac{n + 1}{6} \right) + 5 \left( \frac{n + 2}{6} \right) + 5 \left( \frac{n + 3}{6} \right) + \left( \frac{n + 4}{6} \right). \]
We note that the recursion formula (11) for the $a_{dj}$ can also be derived from (20) using

$$d[xp(d,x)(1 - x)^{d-1}] / dx = p(d + 1,x)(1 - x)^{d-2}.$$  

Next we turn to generalizations of (5) and (12) in which the sequence $0, 1, 2, 3, \cdots$ is replaced by any sequence $U_0, U_1, U_2, U_3, \cdots$ satisfying

$$U_0 = 0, \quad U_1 = 1, \quad U_{n+2} = gU_{n+1} - hU_n \quad \text{for } n = 0, 1, 2, \cdots,$$

and $h^2 = 1$.

The following table indicates some of the well-known sequences that are included for special values of $g$ and $h$:

<table>
<thead>
<tr>
<th>$g$</th>
<th>$h$</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>Natural Numbers: $U_n = n$</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>Fibonacci Numbers: $U_n = F_n$</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>Pell Numbers: $U_n = P_n$</td>
</tr>
<tr>
<td>$L_k$</td>
<td>$(-1)^k$</td>
<td>$U_n = F_k F_n$</td>
</tr>
</tbody>
</table>

A key formula for the generalized sequence $U_n$ is the addition formula

$$U_{2m+n+2} = U_{m+2} U_{n+2} - U_m U_n,$$

which is established by double induction using (25) and verification for the four cases in which $(m,n)$ is $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$.

We now assume that $(g,h)$ is not $(1,1)$ in (25); then (25) is ordinary in the sense of Torretto-Fuchs (see [1]) and so $U_n \neq 0$ for $n > 0$. Then we use the Torretto-Fuchs notation

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ r \end{bmatrix} = \frac{U_m U_{n-1} \cdots U_{n-r+1}}{U_1 U_2 \cdots U_r} \quad \text{for } r = 1, 2, \cdots.$$
for Dov Jarden's generalized binomials. Jarden showed \( \sum \) that

\[
\sum_{j=0}^{d+1} (-1)^j h^j(j-1)/2 \left[ \begin{array}{c} d+1 \\ j \end{array} \right] Z_{n-j} = 0 ,
\]

if \( Z_n \) is the term-by-term product of the \( n \)th terms of \( d \) sequences each of which satisfies the recursion formula (25). The sequence \( Z_n = \left[ \begin{array}{c} n \\ d \end{array} \right] \) is such a product, hence

\[
\sum_{j=0}^{d+1} (-1)^j h^j(j-1)/2 \left[ \begin{array}{c} d+1 \\ j \end{array} \right] \left[ \begin{array}{c} n \\ d \end{array} \right] = 0 .
\]

We are now in a position to give the following generalizations of (5) and (12):

\[
U_n^d = B_d \left[ \begin{array}{c} n + d - 1 \\ d \end{array} \right] + B_{d1} \left[ \begin{array}{c} n + d - 2 \\ d \end{array} \right] + \cdots + B_{d, d-1} \left[ \begin{array}{c} n \\ d \end{array} \right],
\]

where

\[
B_{dj} = \sum_{k=0}^{d-j-1} (-1)^k h^k(k-1)/2 U_{j+1-k}^d \left[ \begin{array}{c} d+1 \\ k \end{array} \right]
\]

Formula (29) is established in the same fashion as for formula (12), with the vanishing of the sums of (16) replaced by (27).

We do not generalize the summation formula (6) since we are not able to give a generalization of formula (1). However, we do present the following summation formulas involving the generalized sequence \( U_n \):

\[
U_2 + U_4 + \cdots + U_{2n} = (U_{n+1}^2 + U_n^2 - U_{n+1}^2)/U_2
\]

\[
U_1 U_3 + U_2 U_6 + \cdots + U_n U_{3n} = U_n U_{n+1} U_{2n+1}/U_2
\]
The formulas are easily probed by mathematical induction using the following special cases of the above addition formula:

\[(33)\quad U_{n+2}^2 - U_n^2 = U_2 U_{2n+2}\]

\[(34)\quad U_{n+2} U_{2n+3} - U_n U_{2n+1} = U_2 U_{3n+3}\]

The special case of (31) in which \(U_n = F_n\) is Recke's problem [3] which brought to mind the well-known formula

\[(35)\quad 1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6.\]

These two special cases inspired the generalization (31). Then (32) was obtained as a generalization of the well-known

\[(36)\quad 1^3 + 2^3 + 3^3 + \cdots + n^3 = n^2(n+1)^2/2.\]

The proofs of (31) and (32) produced (33) as a byproduct; then (30) follows readily using the telescoping sum

\[\sum_{k=1}^{n} [U_2 U_{2k}] = \sum_{k=0}^{n} [U_{k+1}^2 - U_k^2] = U_{n+1}^2 - U_1^2.\]

Some special cases of (28) and a special case of (33) above were proposed by one of the authors [4].

Formulas (5), (11), and (13) go back to J. Worpitzky and G. Frobenius (see [5] and [6]). These have been generalized in a different manner from our formulas (28) and (29) by L. Carlitz [7].

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