PYTHAGORAS REVISITED

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A Pythagorean triplet consists of three numbers (a, b, c) in which $a^2 + b^2 = c^2$. Such triplets are generated by m and n where $m^2 - n^2 = a$, $2mn = b$, and $m^2 + n^2 = c$.

What are the conditions that in such triplets $a + b = L^2$, as in (9, 40, 41)?

Set $m + n = K$ and substitute $K - n = m$ in $a + b = L^2$, letting $a = m^2 - n^2$ and $b = 2mn$. Then, $K^2 - 2n = L^2$ or $K^2 - L^2 = 2n$.

This last equation is of the form $A^2 - B^2 = 2C^2$, whose general solution is $A = t^2 + 2u^2$, $B = t - 2u^2$, and $C = 2tu$ [1].

Hence, $K = t^2 + 2u^2$, $L = t^2 - 2u^2$, and $n = 2tu$. Since $m = K - n$, then by substitution, $m = t^2 + 2u^2 - 2tu$.

We desire to choose $m > n$. This condition will obtain when $t^2 + 2u^2 : tu > 4$.

Several other Pythagorean triplets of this type are (133, 156, 205), (2461, 5460, 5989), and (12,549, 34,540, 36,749).

I. What are the conditions that in Pythagorean triplets $a + b = L^2$ and $m + n = K^2$, and in (1,690,128; 9,412,096; 9,562,640) in which $m = 2372$ and $n = 1984$?

Since $m = K^2 - n$ and the conditions for $a + b = L^2$ have been found above, we can set $t^2 + 2u^2 - 2tu = K^2 - 2tu$. Then $K^2 = t^2 + 2u^2$ or $K^2 - t^2 = 2u^2$, an equation of the form $A^2 - B^2 = 2C^2$. Whence

$$K = x^2 + 2y^2$$
$$t = x^2 - 2y^2$$
$$u = 2xy \quad .$$

Therefore,

$$m = (x^2 - 2y^2)^2 + 2(2xy)^2 - 2(2xy)(x^2 - 2y^2) \ ,$$

or

$$m = (x^2 + 2y^2)^2 - 4xy(x^2 - 2y^2)$$

and

$$n = 4xy(x^2 - 2y^2) .$$

We desire to choose $m > n$. This condition will obtain when $(x^2 + 2y^2)^2 : xy(x^2 - 2y^2) > 8$.

In the example above, $x = 8$, $y = 1$. Another such triplet is one in which $x = 15$ and $y = 2$, $m = 28,249$ and $n = 26,040$.

II. What are the conditions that in Pythagorean triplets $a + b + c = M^2$, as in (63, 16, 65)?

Since $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, we can set

$$m^2 - n^2 + 2mn + m^2 + n^2 = M^2$$

by substitution. Then

$$2m^2 + 2mn - M^2 = 0 .$$

Use the quadratic equation formula to solve for m. Then

$$m = -2n \pm \sqrt{4n^2 + 8M^2} : 4$$

or

$$m = -n \pm \sqrt{n^2 + 2M^2} : 2 .$$

We will show that $n^2 + 2M^2$ is a perfect square when $n = d^2 - 2e^2$ and $M = 2de$.

Set $n^2 + 2M^2 = P^2$. Then

$$P^2 - n^2 = 2M^2,$$

which is an equation of the type $A^2 - B^2 = C^2$. Whence

$$P = d^2 + 2e^2$$
$$n = d^2 - 2e^2$$
$$M = 2de \quad .$$

Then, by substitution,

$$m = -d^2 + 2e^2 \pm \sqrt{(d^2 - 2e^2)^2} + \sqrt{2(2de)^2}:2$$

or $m = 2e^2, \; -e^2$. Discard the negative result.

We desire to choose $m > n$. This condition will obtain when $d < 2e$ and $d^2 > 2e^2$.

In triplets of this type, there is the bonus that $m + n$ is also a square, namely, d^2.

III. What are the conditions that in Pythagorean triplets $a + b = L^2$ and $a^2 = b + c$, as in (57, 1624, 1625)?

Since $a^2 = b + c$, then by substitution,

$$(m^2 - n^2)^2 = 2mn + m^2 + n^2$$

or

$$(m^2 - n^2)^2 = (m + n)^2,$$

whence, $m = n + 1$.

We have shown earlier that if $a + b = L^2$, then $m = t^2 + 2u^2 - 2tu$ and $n = 2tu$. Since $m = n + 1$ if $a^2 = b + c$, then set

$$t^2 - 2tu + 2u^2 = 2tu + 1$$

or

$$t^2 - 4tu + 2u^2 - 1 = 0 \quad .$$

Solve for t using the quadratic equation formula. Then

$$t = 4u \pm \sqrt{16u^2 - 8u^2 + 4}:2$$

or

$$t = 2u \pm \sqrt{2u^2 + 1}.$$

Now $2u^2 + 1$ will be a perfect square when $u = 0, 2, 12, 70, 408, \cdots$, a recurrent series in which $q_1 = 0$, $q_2 = 2$, and $q_n = 6q_{n-1} - q_{n-2}$.

As $u = 0, 2, 12, 70, 408, \cdots$, t correspondingly equals ± 1, 4 ± 3. 24 ± 17, 140 ± 99, $816 \pm 577, \cdots$.

The first six Pythagorean triplets in which $a + b = L^2$ and $a^2 = b + c$ are listed below in abbreviated form, since in these triplets, $n = m - 1$ and $c = B + 1$.

M	A	B
5	9	40
29	57	1,624
169	337	56,784
985	1,969	1,938,480
5,741	11,481	65,906,680
33,461	66,921	2,239,210,120

REFERENCE

1. Albert H. Beiler, Recreations in the Theory of Numbers, Dover Publications, Inc., New York, 1964, p. 129