NUMBERS THAT ARE BOTH TRIANGULAR AND SQUARE THEIR TRIANGULAR ROOTS AND SQUARE ROOTS

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There is an infinite series of numbers, N, which for integral T and S:

(1) $$\tfrac{1}{2}T(T + 1) = N = S^2 .$$

The first nine members of the series are tabulated below, together with their triangular roots, square roots, and index numbers, n.

n	T	N	S
0	0	0	0
1	1	1	1
2	8	36	6
3	49	1225	35
4	288	41616	204
5	1681	1413721	1189
6	9800	48024900	6930
7	57121	1631432881	40391
8	332928	55420693056	235416

By inspection of the tabulation, we note the recursive formula for N:

(2) $$N_n = 34 N_{n-1} - N_{n-2} + 2 ,$$

from which we can develop a generalized formula for N:

(3) $$N_n = \frac{1}{32} \left[(17 + 12\sqrt{2})^n + (17 - 12\sqrt{2})^n - 2 \right] .$$

Similarly,

(4) $$T_n = 7T_{n-1} - 7T_{n-2} + T_{n-3} ,$$

and

(5) $$T_n = \frac{1}{4} [(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n - 2].^*$$

Also:

(6) $$S_n = 6S_{n-1} - S_{n-2} ,$$

and

$$S_n = \frac{1}{8} \sqrt{2} [(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n] .$$

Other recursive formulas and relations were found by inspection of the tabulation:

(7) $$S_{2n} = S_n (S_{n+1} - S_{n-1})$$

(8) $$T_{2n-1} = (T_n - T_{n-1})^2$$

(9) $$S_{2n-1} = N_n - N_{n-1}$$

(10) $$T_{2n} = 8 N_n$$

(11) $$T_n - T_{n-1} = S_n + S_{n-1}$$

(12) $$T_{2n-1} = (S_n + S_{n-1})^2$$

(13) $$S_{2n-1} = (S_n - S_{n-1})(T_n - T_{n-1})$$

(14) $$N_n - N_{n-1} = (S_n - S_{n-1})(T_n - T_{n-1})$$

*This simplification of the author's more complicated formula was furnished by Hoggatt

(15) $$T_{2n} = 8S_n^2$$

(16) $$S_{2n-1} = (S_n - S_{n-1})T_{2n-1}^{\frac{1}{2}}$$

(17) $$N_n - N_{n-1} = (S_n - S_{n-1})(S_n + S_{n-1}) .$$

By the use of the recursive formulas, the tabulation was extrapolated for negative index numbers. It was found to be perfectly reflexive about 0 except that the values of S became negative for negative index numbers, while the values of N and T remained positive. All generalized formulas and recursive formulas and relations held for the reflected series.

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Solution by Using the Fibonacci Terms

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2
 8
 34
 144
  610
  2584
  10946
   46368
   196418
    832040
    ......
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3389......
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$$3 \times 3389\cdots = 1016949\cdots .$$