THE HIDDEN HEXAGON SQUARES

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INTRODUCTION

Pascal's arithmetic triangle has been much studied. Further study continues to produce evidence of the great fertility of this array of numbers. Here we divulge a very surprising result.

Theorem. Let $\binom{m}{n}$ be such that $0 < n < m$, $m \geq 2$, then the product of the six binomial coefficients surrounding $\binom{m}{n}$ is a perfect integer square.

Proof. The six binomial coefficients are:

$$\binom{m-1}{n-1},\ \binom{m-1}{n},\ \binom{m}{n+1},\ \binom{m+1}{n+1},\ \binom{m+1}{n},\ \text{and}\ \binom{m}{n-1}.$$

The product is

$$\frac{(m-1)!}{(n-1)!(m-n)!} \times \frac{(m-1)!}{n!(m-1-n)!} \times \frac{m!}{(n+1)!(m-n+1)!} \times \frac{(m+1)!}{(n+1)!(m-n)!} \times$$

$$\times \frac{m!}{(n-1)!(m-n+1)} = \left[\frac{(m+1)!\,m!\,(m-1)!}{(n-1)!n!(n+1)!(m-n-1)!(m-n)!(m-n+1)!}\right]^2$$

Since each binomial coefficient is an integer, the product is an integer, and since the square of a rational number is an integer if and only if the rational number is an integer, it follows that the product is an integer square.

Corollary. Each alternate triad of the six binomial coefficients have equal products.
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