A DETERMINANT INVOLVING GENERALIZED BINOMIAL COEFFICIENTS

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1. INTRODUCTION

Define the Fibonacci numbers F_n by $F_1 = F_2 = 1$,

$$F_{n+2} - F_{n+1} - F_n = 0 . \tag{1.1}$$

This difference equation may be extended in both directions, yielding

$$F_{-n} = (-1)^{n+1} F_n .$$

Lucas [2] has shown that the $n \times n$ determinant

$$\begin{vmatrix} 1 & 1 & 0 & 0 & \cdots \\ -1 & 1 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 1 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n = F_{n+1} \tag{1.2}$$

This is also a consequence of Problems B-13 [5] and B-16 [6] in this Quarterly. Note that the rows of (1.2) are the negatives of the coefficients of the difference equation (1.1) obeyed by the Fibonacci numbers. The squares of the Fibonacci numbers obey

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0 . \tag{1.3}$$

If we take the negatives of the coefficients of (1.3) and place them in a determinant analogous to (1.2), we find

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$$(1.4)\qquad \begin{vmatrix} 2 & 2 & -1 & 0 & \cdots \\ -1 & 2 & 2 & -1 & \cdots \\ 0 & -1 & 2 & 2 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n = F_{n+1}F_{n+2}$$

Equation (1.4), which appears to be new, may be proved by expanding along the last column and using induction on n. It is our aim to generalize (1.2) and (1.4), first for the Fibonacci sequence, and then for arbitrary second-order recurring sequences.

2. THE FIBONACCI CASE

We define the Fibonacci generalized binomial coefficients $\begin{bmatrix} m \\ r \end{bmatrix}$ by

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \quad (r > 0); \qquad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1 .$$

Note that $\begin{bmatrix} m \\ r \end{bmatrix}$ is defined for all integers and all non-negative integers r, and that

$$(2.1)\qquad \begin{bmatrix} m \\ r \end{bmatrix} = 0 \quad \text{for} \quad m = 0, 1, \cdots, r - 1 .$$

It is convenient to set

$$(2.2)\qquad \begin{bmatrix} m \\ r \end{bmatrix} = 0 \quad \text{for} \quad r < 0 .$$

Jarden [1] showed that the term-by-term product P_n of k - 1 sequences each of which obeys (1.1) satisfies

$$(2.3)\qquad \sum_{j=0}^{k} (-1)^{j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix} P_{n-j} = 0 .$$

In particular, if each is the Fibonacci sequence we have

$$\sum_{j=0}^{k} (-1)^{j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix} F_{n-j}^{k-1} = 0 \; . \tag{2.4}$$

This becomes (1.1) for $k = 2$, and (1.3) for $k = 3$.

Determinants of the form

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & a_0 & a_1 & a_2 & \cdots \\ 0 & 0 & a_0 & a_1 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \tag{2.5}$$

are known as recurrents. We shall put the coefficients of (2.4) into an $n \times n$ recurrent and show its value is yet another generalized binomial coefficient. We remark that a general method for evaluating recurrents, from which the results here would follow, appears to date back to H. Faure (see [3], Vol. 2, p. 212). However, our approach seems somewhat more direct, and the specific results novel enough to warrant separate attention.

Put

$$D_{n,k} = \det (a_{rs}) \; ,$$

where

$$a_{rs} = -(-1)^{(s-r+1)(s-r+2)/2} \begin{bmatrix} k + 1 \\ s - r + 1 \end{bmatrix} \quad (r,s = 1,2,\cdots,n) \; .$$

Recalling (2.1) and (2.2), we see that $D_{n,1}$ is simply (1.2), and that $D_{n,2}$ is (1.4).

For $n > k$, expansion of $\det (a_{rs})$ along the last column and simplification gives

$$D_{n,k} = - \sum_{j=1}^{k+1} (-1)^{j(j+1)/2} \begin{bmatrix} k + 1 \\ j \end{bmatrix} D_{n-j,k} \; . \tag{2.6}$$

If we define

$$D_{0,k} = 1; \quad D_{-n,k} = 0 \quad \text{for} \quad n = 1, 2, \cdots, k-1, \tag{2.7}$$

then (2.6) remains valid for $n \geq 1$. Now for fixed k, $\begin{bmatrix} n \\ k \end{bmatrix}$ is the product of k sequences each obeying (1.1), so that using Jarden's result (2.3) we see

$$\sum_{j=0}^{k+1} (-1)^{j(j+1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix} = 0 . \tag{2.8}$$

By (2.1) and (2.7),

$$D_{n,k} = \begin{bmatrix} n+1 \\ k \end{bmatrix} \quad (n = -k+1, -k+2, \cdots, 0) ,$$

and by (2.6) and (2.8) both $D_{n,k}$ and $\begin{bmatrix} n+k \\ k \end{bmatrix}$ obey the same $(k+1)^{st}$-order recurrence relation. Hence,

$$D_{n,k} = \begin{bmatrix} n+k \\ k \end{bmatrix} . \tag{2.9}$$

Note that this reduces to (1.2) and (1.4) for $k = 1, 2$, respectively.

3. EXTENSION TO SECOND-ORDER RECURRING SEQUENCES

Let the sequence $\{U_n\}$ be defined by $U_0 = 0$, $U_1 = 1$,

$$U_{n+2} - pU_{n+1} + qU_n = 0 \quad (q \neq 0) . \tag{3.1}$$

Let a and b be the roots of the auxiliary polynomial $x^2 - px + q$ of (3.1). We deal only with the case in which (3.1) is ordinary in the sense of R. F. Torretto and J. A. Fuchs [4], i.e., we assume that either $a = b$ or $a^n \neq b^n$ for $n > 0$. It follows that

$$U_n = \begin{cases} \dfrac{a^n - b^n}{a - b} & \text{if} \quad a \neq b\,, \\ na^{n-1} & \text{if} \quad a = b\,. \end{cases}$$

We define the U-generalized binomial coefficients $\left[{m \atop r}\right]_u$ by

$$\left[{m \atop r}\right]_u = \frac{U_m U_{m-1} \cdots U_{m-r+1}}{U_1 U_2 \cdots U_r} \quad (r > 0); \qquad \left[{m \atop 0}\right]_u = 1\,.$$

Note that

$$(3.2) \qquad \left[{m \atop r}\right]_u = 0 \qquad (m = 0, 1, \cdots, r - 1)\,.$$

As with the usual binomial coefficients, we define

$$(3.3) \qquad \left[{m \atop r}\right]_u = 0 \qquad (r < 0)\,.$$

In a generalization of (2.3), Jarden has shown that the term-by-term product Q_n of any $k - 1$ sequences, each obeying (3.1), satisfies

$$(3.4) \qquad \sum_{j=0}^{k} (-1)^j q^{j(j-1)/2} \left[{k \atop j}\right]_u Q_{n-j} = 0\,.$$

Equation (3.4) indeed reduces to (2.3) when $p = -q = 1$. We shall use the negatives of the coefficients of (3.4) to form a recurrent as before.

Let

$$D_{n,k}(U) = \det\,(b_{rs})\,,$$

where

$$b_{rs} = -(-1)^{s-r+1} q^{(s-r)(s-r+1)} \begin{bmatrix} k+1 \\ s-r+1 \end{bmatrix}_u \qquad (r,s = 1,2,\cdots,n) .$$

We find it convenient to set

$$(3.5) \qquad D_{0,k}(U) = 1; \quad D_{-n,k}(U) = 0 \qquad (n = 1, 2, \cdots, k-1) .$$

Then expansion of $\det(b_{rs})$ along the last column gives

$$(3.6) \qquad D_{n,k}(U) = -\sum_{j=1}^{k+1} (-1)^j q^{j(j-1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix}_u D_{n-j,k}$$

for all $n \geq 1$.

Noticing that $\begin{bmatrix} n \\ k \end{bmatrix}_u$ is the product of k sequences eacy obeying (3.1), we see from (3.4) that

$$(3.7) \qquad \sum_{j=0}^{k+1} (-1)^j q^{j(j-1)/2} \begin{bmatrix} k+1 \\ j \end{bmatrix}_u \begin{bmatrix} n-j \\ k \end{bmatrix}_u = 0 .$$

Then

$$D_{n,k}(U) = \begin{bmatrix} n+k \\ k \end{bmatrix}_u \qquad (n = -k+1, -k+2, \cdots, 0) ,$$

and by (3.6) and (3.7), $D_{n,k}(U)$ and $\begin{bmatrix} n+k \\ k \end{bmatrix}_u$ obey the same $(k+1)^{st}$-order recurrence relation. Hence,

$$(3.8) \qquad D_{n,k}(U) = \begin{bmatrix} n+k \\ k \end{bmatrix}_u .$$

We conclude by investigating some particular cases of (3.8). First note that it reduces to (2.9) for $p = -q = 1$. If

$$p = L_s = F_{s-1} + F_{s+1}, \qquad q = (-1)^s ,$$

then $U_n = F_{sn}$, so that for $k = 2$, (3.8) yields

$$\begin{vmatrix} L_s & (-1)^{s+1} & 0 & 0 & \cdots \\ -1 & L_s & (-1)^{s+1} & 0 & \cdots \\ 0 & -1 & L_s & (-1)^{s+1} & \cdots \\ 0 & 0 & -1 & L_s & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n = F_{s(n+1)} \,.$$

Putting $s = 1$ proves (1.2).

If we let $p = 2$, $q = 1$, then $a = b = 1$ and $U_n = n$. In this case,

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \binom{m}{r},$$

the usual binomial coefficient. Equation (3.8) then yields

$$\det\left[-(-1)^{s-r+1}\binom{k+1}{s-r+1}\right] = \binom{n+k}{k} \qquad (r, s = 1, 2, \cdots, n)\,.$$

In particular, for $k = 2$, we find

$$\begin{vmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & \cdots \\ 0 & -1 & 2 & -1 & \cdots \\ 0 & 0 & -1 & 2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_n = n + 1\,,$$

which first seems to have been noted by Welstenholme (see [3], Vol. 3, p. 394). Letting $k = 3$, we obtain

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