ON THE GENERATION OF FIBONACCI NUMBERS AND THE "POLYVIBRATING" EXTENSION OF THESE NUMBERS

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SUMMARY

The authors, while emphasizing the role of Fibonacci numbers in the elaboration of the method of sequential search for the optimum, exhibit new generations of these numbers, taking as the point of departure either a certain ordinary, second-order differential equation, or a minimization problem of a certain functional. Then they present, while continuing their studies concerning polyvibrating systems and generalized polyvibratings, a "polyvibrating" extension of Fibonacci numbers.

1. Developments concerning the method of sequential search for the determination of the optimum, having at its basis Fibonacci numbers, have motivated in the recent times the work which refers itself so much to methods of finding the optimum in domains of several dimensions, as well as to different extensions of Fibonacci numbers. It is in the framework of this admirable progress which we helped in founding seven years ago and to the regular publication since then of a periodical specializing in this area, namely THE FIBONACCI QUARTERLY.

In the following, the authors give new generations of Fibonacci numbers, taking as the point of departure either a certain second-order ordinary differential equation, or a problem of minimizing a certain functional. They finally present, while continuing their research concerning polyvibrating systems and generalized polyvibratings, a "polyvibrating" extension of Fibonacci numbers.

2. Given

$$(a_1x + a_2)y'' + (a_3x + a_4)y' + (a_5x + a_6)y = 0 , \tag{2.1}$$

the second-order differential equation of Laplace. The recurrence relation between the coefficients K_i of one of the equations solutions is written in the form of a power series expansion, which is given by

$$\frac{a_2}{n+2} K_{n+1} = -\frac{a_1 n + a_4 - a_1}{n+1} K_n - \frac{na_3 + a_6 - a_3}{n} K_{n-1} - a_5 K_{n-2} \tag{2.2}$$

and the first terms of the set of coefficients determined by (2.1) are

$$\begin{aligned} &1;\ 1;\ -\frac{3}{a_2}\left(\frac{a_4}{2} + a_6\right);\ \frac{4}{a_2^2}\left[(a_1 + a_2)\left(\frac{1}{2} a_4 + a_6\right) - \frac{3}{2}(a_3 + a_6) - 3a_5\right];\\ &-\frac{20}{a_2^3}(2a_1 + a_4)\left[(a_1 + a_4)\left(\frac{1}{2} a_4 + a_6\right) - \frac{3}{2}(a_3 + a_6) - 3a_5\right] + \\ &\qquad \frac{5(2a_3 + a_6)}{a_2^2} - \frac{5a_5}{a_2};\ \cdots \ . \end{aligned} \tag{2.3}$$

The direct calculation leads to the following theorem:

Theorem 1. Equation (2.1) reduces for

$$a_1 = 1, \quad a_2 = 0, \quad a_3 = -1, \quad a_4 = 2, \quad a_5 = -1, \quad a_6 = -1 \tag{2.4}$$

to the differential equation whose solution, in the form of a power series expansion, have for coefficients the Fibonacci numbers.

3. Following the fact that the equation of Laplace (2.1) results in the problem of minimizing the functional

$$\begin{aligned} [y(x)] = \int_a^b \Bigg[&-\frac{a_3 x^2 + 2a_4 x - 2a_1 x}{2}\left(yy'' + \frac{1}{2} y'^2\right) \\ &+ \frac{a_5 x + a_6}{2} y^2 - \frac{a_1 x + a_2 y'^2}{2}\Bigg] dx \end{aligned} \tag{3.1}$$

in the set of functions satisfying the conditions

$$y(a) = y(b) = 0 \,, \tag{3.2}$$

results in the following theorem.

Theorem 2. The minimization of the functional in the set of

$$T[y(x)] = \int_a^b \left[- \frac{-x^2 + 2x}{2} \; yy'' + \frac{y'^2}{2} - \frac{x+1}{2} y^2 - \frac{x}{2} y'^2 \; dx \right] \tag{3.3}$$

functions $y(x)$ satisfying (3.2) leads to the second-order ordinary differential equation

$$xy'' + (2 - x)y' - (x + 1)y = 0 \,, \tag{3.4}$$

of which the successive coefficients of the solution expressed as a power series represent the sequence of Fibonacci numbers.

4. Now consider the polyvibrating extension of the sequence of Fibonacci numbers taking the point of departure of polyvibrating systems, of which the prototype is given by the eigenvalue problem

$$D[A(x)Du + \lambda B(x)u] + \lambda[B(x)Du + C(x)u] = 0, \quad u\Big|_{FrR} = 0 \,, \tag{4.1}$$
$$x = (x_1, x_2, x_3, \cdots, x_m) \,,$$

whose novelty consists in taking the rectangular domain

$$R = \{a_i \le x_i \le b_i; \quad i = 1, 2, 3, \cdots, m\}$$

which has m dimensions and the symbol D denoting the polyvibrating derivative, or (better) the total derivative in the sense of M. M. Picone, namely

$$\text{(4.2)} \qquad Du \equiv \frac{\partial^m u}{\partial x_1 \partial x_2 \cdots \partial x_m}$$

or (better) be the variational problem

$$\text{(4.3)} \qquad G[f(x)] = \int_R A(x)\,[Df(x)]^2 dx, \qquad (x = (x_1, x_2, \cdots, x_m)\ ,$$

$$R = \{a_i \le x_i \le b_i\}, \qquad (i = 1, 2, \cdots, m)$$

and

$$\text{(4.4)} \qquad H[f(x)] = \int_R [2B(x)f(x)Df(x) + C(x)f^2(x)]dx = \pm 1, \quad f(x)\Big|_{FrR} = 0 .$$

In the case where one considers the polyvibrating equation of Laplace

$$\text{(4.5)} \quad (a_1\theta + a_2)D^2u + (a_3\theta + a_4)Du + (a_5\theta + a_6)u = 0, \quad \theta = \prod_{i=1}^{m} x_i \ ,$$

the recurrence relation generating the coefficients in the form of a power series solution of the product $\theta = x_1, x_2, \cdots, x_m$ of the equation (4.5) is

$$\text{(4.6)} \qquad \begin{aligned} a_2(n+1)^m n^m k_{n+1} &= -\left[a_1 n^m [n-1]^m + a_4 n^m\right](n+2)k_n - (n+1)(n+2) \\ &\cdot [a_3(n-1)^m + a_6]k_{n-1} - a_5 n(n+1)(n+2)k_{n-2} \ , \end{aligned}$$

while the relation generating the polyvibrating extension of Fibonacci numbers (the hypothesis (2.4) concerning the coefficients of equation (4.5) is given by

$$\text{(4.7)} \qquad [n^m(n = 1)^m + 2n^m]k_n = (n+1)\,[(n-1)^m + 1]k_{n-1} + n(n+1)k_{n-2}$$

and the first terms of corresponding sequence are

$$(4.8) \qquad 1;\ 2^{1-m};\ \frac{4}{3^m(2^m + 2)}\left[\frac{2^m + 1}{2^m - 1} + 3\right];\ \cdots \quad .$$

REMARKS

1. It would be interesting to give a geometric interpretation for the coefficients of the sequence (2.3) and of its polyvibrating extension (4.8).

2. The application to the variational problem (3.2), (3.3) and to its polyvibrating extension (4.3) and (4.4) of the method of dynamic programming or by other present methods of optimization could perhaps clear up the (why) of the fundamental role which is performed by the sequence of Fibonacci numbers and the corresponding differential equation in sequential search, which is used with such success in the theory of supplies of all kinds, automatic sample control genetics, separation processes of separation of several phases and still others.

3. We refer finally for algorithmic details and other results concerning this class of ideas to the paper in Bulletin of Polytechnic Institute of Jassey. To be found there among other ideas, are the extensions of the Fibonacci numbers and the many relations that connect them, corresponding to generalized polyvibrating systems, that is to the systems of the form (2.1), (2.4), where the ordinary differential operator d/dx and the independent variable x are systematically and respectively replaced by the generalized polyvibrating operator 16 - 17.

$$D^*u = \frac{\partial^{n_1+n_2+\cdots+n_m}}{\partial x^{n_1}\,\partial x_2^{n_2}\cdots\partial x_m^{n_m}}$$

and by the product of independent variables

$$(5.2) \qquad \theta^* = \prod_{i=1}^{m} x_i^{n_i} \quad .$$

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