RELATIONS BETWEEN A SEQUENCE OF FIBONACCI TYPE AND THE SEQUENCE OF ITS PARTIAL SUMS

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Let $\{F_n\}$ be a Fibonacci-type sequence, where

$$F_{n+2} = F_n + F_{n+1}, \quad F_1 = a, \quad F_2 = b \ .$$

Let $\{S_n\}$ be the sequence obtained from the given sequence by taking the partial sums of its terms, that is,

$$S_n = \sum_{i=1}^{n} F_i \ .$$

Then there is an unexpected relation between the F-sequence and the S-sequence, namely that S_{4r+2} is a multiple of F_{2r+1}. In fact,

$$S_{4r-2} = c_{2r-1} F_{2r+1} \ ,$$

where $\{c_n\}$ is itself a Fibonacci-type sequence with $c_1 = 1$ and $c_2 = 3$ (the Lucas sequence).

<u>Proof</u>. If α and β are the roots of the equation

$$t^2 - t - 1 = 0 \ ,$$

we may show, as usual, that

$$F_n = \frac{a(\alpha^{n-2} - \beta^{n-2}) + b(\alpha^{n-1} - \beta^{n-1})}{\alpha - \beta}$$

We may then obtain S_n by summing geometric progressions, which gives

$$S_n = \frac{a(\alpha^n - \beta^n) + b[\alpha(\alpha^n - 1) - \beta(\beta^n - 1)]}{\alpha - \beta} .$$

Hence,

$$F_{2r+1} = \frac{a(\alpha^{2r-1} - \beta^{2r-1}) + b(\alpha^{2r} - \beta^{2r})}{\alpha - \beta}$$

and

$$S_{4r-2} = \frac{a(\alpha^{4r-2} - \beta^{4r-2}) + b[\alpha(\alpha^{4r-2} - 1) - \beta(\beta^{4r-2} - 1)]}{\alpha - \beta} .$$

Since

$$\begin{aligned}(\alpha^{2r} - \beta^{2r})(\alpha^{2r-1} + \beta^{2r-1}) &= \alpha^{4r-1} - \beta^{4r-1} + (\alpha\beta)^{2r-1}(\alpha - \beta) \\ &= \alpha^{4r-1} - \beta^{4r-1} - (\alpha - \beta) ,\end{aligned}$$

we have

$$S_{4r-2} = c_{2r-1} F_{2r+1} ,$$

where

$$c_{2r-1} = \alpha^{2r-1} + \beta^{2r-1} .$$

It follows that $\{c_n\}$ is itself a Fibonacci-type sequence, with

$$c_1 = \alpha + \beta = 1$$

and

$$c_2 = \alpha^2 + \beta^2 = 3 .$$

There are two other results of interest. First, we have a somewhat similar relation between S_{4r} and F_{2r+2}, namely

$$S_{4r} = c_{2r}F_{2r+2} - 2b \; .$$

This can easily be proved in the same way as the earlier result.

Second, it follows from the earlier results that

$$S_n = F_{n+2} - b \; ,$$

and hence that

$$S_{n+2} = S_n + S_{n+1} + b \; .$$

[Continued from page 295.]

— for (5^{bis}), the product AB by the product of the values of A and B relative to the sequences y and z, and, on the other hand, ϕ by the sum of the values of ϕ for these sequences.

— for (6^{bis}) $\sqrt{AB}$ by AB and ϕ by 2ϕ ,

— for (7^{bis}) $\sqrt{AB}$ by AB and ϕ by $p + 2\phi$.

12. The author thinks he has shown, by the present study which does not maintain to be exhaustive, how much the use of hyperbolic lines to express the terms of the linear sequences of the type (1) is favorable by the simplicity which it introduces in the calculations bearing on these sequences, by the fact also that it suggests relations, which makes it easier to set up. These advantages are specially clear in the case of the Fibonacci and Lucas sequences, for which it is possible to re-establish quickly the well known formulas concerning them.