RECIPROCALS OF GENERALIZED FIBONACCI NUMBERS

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1. INTRODUCTION

The purpose of this paper is to find expressions for

$$\sum_{n=1}^{\infty} H_{2n}^{-1}, \qquad \sum_{n=1}^{\infty} H_n^{-t} z^n, \qquad H_n^{-1} \quad \text{and} \quad H_{n+1}^{-t},$$

where $\{H_n\}$ is the generalized Fibonacci sequence defined by Horadam [6] as follows:

$$(1.1) \qquad H_n = H_{n-1} + H_{n-2} \quad (n \geq 3), \qquad H_1 = p, \qquad H_2 = p + q,$$

where p, q are arbitrary integers, and

$$(1.2) \qquad H_n = (2\sqrt{5})^{-1} (\ell a^n - m b^n)$$

with $\ell = 2(p - qb)$, $m = 2(p - qa)$ and where a, b are the roots of $x^2 - x - 1 = 0$.

The required expressions will be obtained as results (2.1), (2.2), (2.3), and (3.6), respectively. They will be seen to involve Lambert series and Bernoulli-type polynomials.

Let

$$(1.3) \qquad H = \frac{p - qb}{p - qa}.$$

We define the Lambert series

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$$L_1(x) = \sum_{r=1}^{\infty} H^{-r/2} \frac{x^r}{1 - x^r} \tag{1.4}$$

and

$$L_2(x) = \sum_{r=1}^{\infty} H^{-r} \frac{x^r}{1 - x^r} \ . \tag{1.5}$$

Details of some of the properties of the Lambert series may be found in Hardy and Wright [5] and Landau [7].

We also need to introduce a new expression

$$\sum_{r=0}^{\infty} B_r^{(t)'}(x) \frac{n^r}{r!} = \frac{n^t e^{nx}}{(e^n - H)^t} \ , \tag{1.6}$$

in which the $B_r^{(t)'}(x)$ is analogous to the general Bernoulli polynomials of higher order which have been discussed by Gould [3].

A Bernoulli polynomial $B_r(x)$ is defined by means of

$$\sum_{r=0}^{\infty} B_r(x) \frac{n^r}{r!} = \frac{n e^{nx}}{e^n - 1} \ . \tag{1.7}$$

Some of their properties are developed by Carlitz [2], Hardy and Wright [5], and Gould [3] and [4] who relates the Bernoulli and Euler numbers.

In fact, the $B_r^{(t)'}(x)$ are generalized Bernoulli polynomials and satisfy the recurrence relation

$$B_r^{(t)'}(x + 1) - H B_r^{(t)'}(x) - n B_r^{(t-1)'}(x) = 0 \ . \tag{1.8}$$

The proof of (1.8) is as follows.

$$\sum_{r=0}^{\infty} \{B_r^{(t)'}(x+1) - HB_r^{(t)'}(x)\} \frac{n^r}{r!}$$

$$= \frac{n^t e^{nx} e^n}{(e^n - H)^t} - \frac{Hn^t e^{nx}}{(e^n - H)^t}$$

$$= n \frac{n^{t-1} e^{nx}}{(e^n - H)^{t-1}} = n \sum_{n=0}^{\infty} B_r^{(t-1)'}(x) \frac{n^r}{r!}$$

We shall also use a special case of $B_r^{(t)'}(x)$, when $r = 1$, defined by

(1.9)
$$\sum_{r=0}^{\infty} B'_r(x) \frac{n^r}{r!} = \frac{ne^{nx}}{e^n - H} .$$

The $B'_r(x)$ also satisfy a recurrence relation

$$B'_r(x+1) - HB'_r(x) = rx^{r-1} .$$

This recurrence relation follows since

$$\sum_{r=0}^{\infty} \{B'_r(x+1) - HB'_r(x)\} \frac{n^r}{r!}$$

$$= \frac{ne^{nx} e^n}{e^n - H} - \frac{Hne^{nx}}{e^n - H}$$

$$= ne^{nx} = n \sum_{r=0}^{\infty} \frac{(nx)^r}{r!} .$$

2. CALCULATION OF THE RECIPROCALS

$$\sum_{n=1}^{\infty} H_{2n}^{-1} = 2\sqrt{5}\sum_{n=1}^{\infty} \frac{1}{\ell a^{2n} - mb^{2n}}$$

$$= 2\sqrt{\frac{5}{\ell m}}\sum_{n=1}^{\infty} \frac{\sqrt{\frac{m}{\ell}}\, b^{2n}}{1 - \frac{m}{\ell} b^{4n}}$$

$$= 2\sqrt{\frac{5}{\ell m}}\sum_{n=1}^{\infty} \left\{ \frac{\frac{1}{\sqrt{H}}\, b^{2n}}{1 - \frac{1}{\sqrt{H}}\, b^{2n}} - \frac{\frac{1}{H} b^{4n}}{1 - \frac{1}{H} b^{4n}} \right\}$$

$$= 2\sqrt{\frac{5}{\ell m}}\sum_{n=1}^{\infty}\sum_{r=1}^{\infty} \left\{ H - \frac{r}{2}\, b^{2nr} - H^{-r} b^{4nr} \right\}$$

$$= 2\sqrt{\frac{5}{\ell m}}\sum_{r=1}^{\infty} \left\{ H^{-\frac{r}{2}}\, \frac{b^{2r}}{1 - b^{2r}} - H^{-r}\, \frac{b^{4r}}{1 - b^{4r}} \right\} .$$

Thus

$$(2.1) \qquad \sum_{n=1}^{\infty} H_{2n}^{-1} = 2\sqrt{\frac{5}{\ell m}} \left(L_1\left(\frac{3 - \sqrt{5}}{2}\right) - L_2\left(\frac{7 - 3\sqrt{5}}{2}\right)\right)$$

That is, the required expression is seen to involve Lambert series defined in (1.4) and (1.5).

Write

$$H_n^{-t} = \left(\frac{2\sqrt{5}}{-m}\right)^t \cdot \frac{1}{a^{nt}} \cdot \frac{1}{(C^n - H)^t} ,$$

where $C = b/a$.

Then

$$H_n^{-t} = \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(C^x a^t)^n} \frac{C^{nx}}{(C^n - H)^t}$$

$$= \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(C^x a^t)^n} \frac{e^{x(n \log C)}}{(e^{n \log C} - H)^t}$$

$$= \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(n \log C)^t (C^x a^t)^n} \frac{z^t e^{xz}}{(e^n - H)^t}$$

where $z = n \log C$. Thus

$$H_n^{-t} = \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(n \log C)^t (C^x a^t)^n} \sum_{r=0}^{\infty} B_r^{(t)'}(x) \frac{(n \log C)^r}{r!}$$

$$(\alpha) \qquad = \left(\frac{2\sqrt{5}}{-m}\right)^t \frac{1}{(C^x a^t)^n} \sum_{r=0}^{\infty} B_r^{(t)'}(x) \frac{(\log C)^{r-t}}{r!} n^{r-t} .$$

From this, the generating function for powers of the reciprocals can be set up. This is

$$(2.2) \quad \sum_{n=1}^{\infty} H_n^{-t} z^n = \left(\frac{2\sqrt{5}}{-m}\right)^t \sum_{r=0}^{\infty} B_r^{(t)'}(x) \frac{(\log C)^{r-t}}{r!} \cdot \sum_{n=1}^{\infty} n^{r-t} \left(\frac{z}{a^{t-x} b^x}\right)^n .$$

Thus, the required expression involves the generalized Bernoulli polynomials of higher order (1.6).

As a special case of (α) with $t = 1$, it follows that

$$(2.3) \qquad H_n^{-1} = \frac{-2\sqrt{5}}{m(a^{1-x} b^x)^n} \sum_{r=0}^{\infty} B'_r(x) \frac{(\log C)^{r-1}}{r!} n^{r-1} .$$

As expected from (2.2), our expression involves the Bernoulli polynomials (1.9).

Following Gould [3], let

$$H(x) = \sum_{n=1}^{\infty} H_n^{-1} x^n . \tag{2.4}$$

Then

$$\ell H(ax) - mH(bx) = \sum_{n=1}^{\infty} H_n^{-1} \left(\frac{\ell a^n - mb^n}{2\sqrt{5}} \right) 2\sqrt{5}\, x^n$$

$$= \sum_{n=1}^{\infty} 2\sqrt{5}\, x^n .$$

Thus

$$\ell H(ax) - mH(bx) = \frac{2\sqrt{5}\,x}{1 - x} , \tag{2.5}$$

which is a succinct expression involving

$$\sum_{n=1}^{\infty} H_n^{-1} x^n .$$

3. THE OPERATOR E

We introduce an operator E, defined by

$$EH_n = H_{n+1} . \tag{3.1}$$

Thus

$$H_{n+2} - H_{n+1} - H_n = 0$$

becomes

$$(E^2 - E - 1)H_n = 0$$

or

(3.2) $$(E - a)(E - b)H_n = 0 .$$

Let

$$G_n = (E - b)H_n = H_{n+1} - bH_n .$$

Then from (3.2),

(3.3) $$(E - a)G_n = 0 \qquad \text{or} \qquad G_{n+1} = aG_n ,$$

and so

(3.4) $$G_1 = H_2 - bH_1 = ap + q .$$

It follows from (3.3) and (3.4) that

(3.5) $$G_n = a^{n-1}(ap + q) .$$

Now

$$H_{n+1} = bH_n + G_n ,$$

and so

$$H_{n+1}^{-t} = b^{-t} H_n^{-t} \left(1 + \frac{G_n}{bH_n}\right)^t$$

$$= b^{-t} H_n^{-t} \sum_{r=0}^{\infty} (-1)^r \frac{t(t+1)\cdots(r+r-1)}{r!} \left(\frac{G_n}{bH_n}\right)^r$$

$$= b^{-t} H_n^{-t} \sum_{r=0}^{\infty} \frac{(-1)^r (t)_r a^{nr-r}}{r!\, b^r H_n^r} (ap+q)^r \qquad ,$$

where

$$(t)_r = t(t+1)(t+2)\cdots(t+r-1) \quad [1] .$$

Thus

$$H_{n+1}^{-t} = \sum_{r=0}^{\infty} \frac{(-1)^r (t)_r}{r!} \sum_{s=0}^{\infty} \frac{r!}{s!\, r-s!} \frac{p^{r-s} q^s}{a^{s-nr} b^{r+t}} H_n^{-t-r}$$

and so

$$(3.6) \qquad H_{n+1}^{-t} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r (t)_r}{s!\, r-s!} \frac{p^{r-s} q^s}{a^{s-nr} b^{r+t}} H_n^{-t-r} .$$

See also (α).

We have thus established expressions for the reciprocals stated at the beginning of this article.

REFERENCES

1. L. Carlitz, "Some Orthogonal Polynomials Related to Fibonacci Numbers," Fibonacci Quarterly, Vol. 4, 1966, pp. 43-48.
2. L. Carlitz, "Bernoulli Numbers," Fibonacci Quarterly, Vol. 6, 1968, pp. 71-85.
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