ADVANCED PROBLEMS AND SOLUTIONS

Edited By
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-183 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California.

Consider the display indicated below.

$$
\begin{array}{ccccccc}
1 \\
1 & 1 \\
2 & 2 & 1 \\
5 & 4 & 3 & 1 \\
13 & 9 & 7 & 4 & 1 \\
34 & 22 & 16 & 11 & 5 & 1 \\
89 & 56 & 38 & 27 & 16 & 6 & 1 \\
 & & & \cdots
\end{array}
$$

(i) Find an expression for the row sums.

(ii) Find a generating function for the row sums.

(iii) Find a generating function for the rising diagonal sums.

H-184 Proposed by Raymond E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Define the cycle α_n $(n = 1, 2, \cdots)$ as follows:

(1) $\alpha_n = (1234 + \cdots + F_n)$, where F_n denotes the n^{th} Fibonacci number.

Now construct a sequence of permutations

$$\left\{\alpha_n^{F_i}\right\}_{i=1}^{\infty}, \qquad (n = 1, 2)$$

where

$$\text{(ii)} \qquad \alpha_n^{F_{i+2}} = \alpha_n^{F_i} \cdot \alpha_n^{F_{i+1}} \qquad (i \geq 1) .$$

Finally, define a sequence $\{u_n\}_{n=1}^{\infty}$ as follows:
u_n is the period of (ii); i.e., u_n is the smallest positive integer such that

$$\text{(iii)} \qquad \alpha_n^{F_{i+u_n}} = \alpha_n^{F_i} \qquad (i \geq N) .$$

a. Find a closed-form expression for u_n.

b. If possible, show $N = 1$ is the minimum positive integer for which (iii) holds for all n.

H-185 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$(1 - 2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+k}{2k} \binom{2k}{k} (1 - v)^{n-k} {}_2F_1[-k, n + k + 1; k + 1; x] ,$$

where ${}_2F_1[a, b; c; x]$ denotes the hypergeometric function.

SOLUTIONS

H-127 REVISITED

H-164 Proposed by Murray S. Klamkin, Ford Motor Company, Dearborn, Michigan.

Generalize H-127 and find a recurrence relation for the product

$$C_n = A_n(x)\,B_n(y),$$

where A_n and B_n satisfy the general second-order recurrence equations:

$$(1) \qquad A_{n+1}(x) = R(x)\,A_n(x) + S(x)\,A_{n-1}(x)$$

$$(2) \qquad B_{n+1}(y) = P(y)\,B_n(y) + Q(y)\,B_{n-1}(y),$$

$a \geq 1$ and A_0, A_1, B_0, B_1 arbitrary.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

We consider the following more general situation. Let E denote the operator defined by $Ef(n) = f(n+1)$. Let $\alpha_1, \cdots, \alpha_r$, $\beta_1, \cdots, \beta_s$ denote $r + s$ arbitrary constants and assume that

$$(1) \qquad (E - \alpha_1) \cdots (E - \alpha_r)\,A_n = 0$$

$$(2) \qquad (E - \beta_1) \cdots (E - \beta_s)\,B_n = 0\,.$$

If $C_n = A_n B_n$, we shall show that

$$(3) \qquad \prod_{i=1}^{r}\prod_{j=1}^{s} (E - \alpha_i\beta_j) \cdot C_n = 0\,.$$

If the α's are distinct and the β's are distinct, the proof of this assertion is easy. In this case, the general solution of (1) is given by

$$A_n = c_1\alpha_1^n + \cdots + c_r\alpha_r^n\,,$$

where $c_1, \cdots, c_r$ are independent of n; the general solution of (2) is

$$B_n = d_1\beta_1^n + \cdots + d_s\beta_s^n\,,$$

where $d_1, \cdots, d_s$ are independent of n. Then

$$C_n = c_1 d_1 (\alpha_1 \beta_1)^n + \cdots + c_r d_s (\alpha_r \beta_s)^n$$

and (3) follows at once.

For the general case we require the following lemma. Let

$$(E - \alpha)^r A_n = 0, \qquad (E - \beta)^s B_n = 0 .$$

Then $C_n = A_n B_n$ satisfies

$$(E - \alpha\beta)^{r+s-1} C_n = 0.$$

To prove this, note that

$$A_n = P_{r-1}(n)\alpha^n ,$$

where $P_{r-1}(n)$ is a polynomial in n of degree $r - 1$ with arbitrary constant coefficients:

$$B_n = Q_{a-1}(n)\, \beta^n ,$$

where $Q_{s-1}(n)$ is a polynomial in n of degree $s - 1$ with arbitrary constant coefficients. Then

$$C_n = P_{r-1}(n)\, Q_{s-1}(n)\, (\alpha\beta)^n$$

and the assertion follows at once.

Now let

$$(E - \alpha_1)^{e_1} \cdots (E - \alpha_r)^{e_r} A_n = 0$$

$$(E - \beta_1)^{f_1} \cdots (E - \beta_s)^{f_s} B_n = 0,$$

where the α's and β's are distinct. Then, by the lemma,

$$\prod_{i=1}^{r}\prod_{j=1}^{s}(E - \alpha_i\beta_j)^{e_i+f_j-1} \cdot C_n = 0 . \tag{4}$$

This result is somewhat stronger than (3). The degree of the operator in the left member of (4) is equal to

$$\sum_{i=1}^{r}\sum_{j=1}^{s}(e_i + f_j - 1) = s\sum_{i=1}^{r} e_i + r\sum_{j=1}^{s} f_j - rs .$$

When some of the α's and β's are equal, c_n may satisfy a recurrence of even lower degree. For example, if

$$(E - \alpha_1)\cdots(E - \alpha_r)A_n = 0,$$

$$(E - \alpha_1)\cdots(E - \alpha_r)B_n = 0 ,$$

then C_n satisfies

$$(E - \alpha_1^2)(E - \alpha_1\alpha_2)\cdots(E - \alpha_r^2)C_n = 0 ,$$

a recurrence of order $n(n + 1)/2$.

Also solved by C. B. A. Peck, M. Yoder, and the Proposer.

SHORT-TERM INDUCTION

H-165 Proposed by H. H. Ferns, Victoria, B.C., Canada.

Prove the identity

$$\sum_{i=1}^{n}\binom{n}{i}\frac{F_{ki}}{F_{k-2}^{i}} = \left(\frac{F_k}{F_{k-2}}\right)^{n} F_{2n} \qquad (k \neq 2) ,$$

where F_i denotes the i^{th} Fibonacci number.

Solution by the Proposer.

The proof of the given identities is based on the two identities:

$$F_{n-2} + \alpha^n = \alpha^2 F_n \tag{1}$$

$$F_{n-2} + \beta^n = \beta^2 F_n \quad , \tag{2}$$

in which $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. These are readily proved by induction on n. Thus, if in (1), we put $n = 1$, we get

$$F_{-1} + \alpha = \alpha^2 F_i$$

or

$$1 + \alpha = \alpha^2 \quad ,$$

which is true. Assuming that (1) is true for $n = 1, 2, \cdots, r = 1, r$, we have

$$F_{r-3} + \alpha^{r-1} = \alpha^2 F_{r-1}$$

and

$$F_{r-2} + \alpha^r = \alpha^2 F_r \quad .$$

Adding corresponding members of these two equations, we get

$$\begin{aligned} F_{r-3} + F_{r-2} + \alpha^{r-1} + \alpha^r &= \alpha^2 (F_{r-1} + F_r) \\ F_{r-1} + \alpha^{r-1}(1 + \alpha) &= \alpha^2 F_{r+1} \\ F_{r-1} + \alpha^{r+1} &= \alpha^2 F_{r+1} \quad . \end{aligned}$$

Hence the induction is complete for the proof of (1). The proof of (2) is similar.

Continuing with the proof of the given identity, we have from (1)

$$\left(1 + \frac{\alpha^k}{F_{k-2}}\right)^n = \left(\frac{F_k}{F_{k-2}}\alpha^2\right)^n \qquad (k \neq 2) \quad .$$

Hence

$$\sum_{i=0}^{n} \binom{n}{i}\left(\frac{\alpha^k}{F_{k-2}}\right)^i = \left(\frac{F_k}{F_{k-2}}\right)^n \alpha^{2n} \quad . \tag{3}$$

In a similar manner (2) yields

$$\sum_{i=0}^{n} \binom{n}{i}\left(\frac{\beta^k}{F_{k-2}}\right)^i = \left(\frac{F_k}{F_{k-2}}\right)^n \beta^{2n} \quad . \tag{4}$$

Subtracting members of (4) from the corresponding members of (3) we have

$$\sum_{i=1}^{n} \binom{n}{i} \frac{F_{ki}}{F_{k-2}^i} = \left(\frac{F_k}{F_{k-2}}\right)^n F_{2n} \qquad (k \neq 2) \quad .$$

This completes the proof of the given identity.

Note that addition of (3) and (4) yields

$$\sum_{i=1}^{n} \binom{n}{i} \frac{L_{ki}}{F_{k-2}^i} = \left(\frac{F_k}{F_{k-2}}\right)^n L_{2n} - 2 \qquad (k \neq 2) \quad .$$

Some special cases are of interest. Putting $k = 1$ and $k = 3$ in these two identities, we get the following.

$$\sum_{i=1}^{n} \binom{n}{i} F_i = F_{2n}, \quad \sum_{i=1}^{n} \binom{n}{i} F_{3i} = 2^n F_{2n}$$

$$\sum_{i=1}^{n} \binom{n}{i} L_i = L_{2n} - 2, \quad \sum_{i=1}^{n} \binom{n}{i} L_{3i} = 2^n L_{2n} - 2 .$$

Also solved by A. Shannon, M. Yoder, C. B. A. Peck, L. Carlitz, and D. V. Jaiswal.

SUM EVEN INDEX

H-166 Proposed by H. H. Ferns, Victoria, B.C., Canada (Corrected).

Prove the identity

$$F_{2mn} = \begin{cases} \sum_{i=1}^{n} \binom{n}{i} L_m^i F_{mi}, & \text{if } m \text{ is odd} \\ \sum_{i=1}^{n} (-1)^{n+i} \binom{n}{i} L_m^i F_{mi}, & \text{if } m \text{ is even .} \end{cases}$$

Solution by the Proposer.

In the identity (this Journal, Vol. 7, No. 2, p. 174),

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda, \qquad (m \neq k) ,$$

put $\lambda = 0$ and $k = 2m$. We get

$$\sum_{i=1}^{n} \binom{n}{i}\left(\frac{F_{2m}}{F_{-m}}\right)^i F_{mi} = \left(\frac{F_m}{F_{-m}}\right)^n F_{2mn}$$

$$\sum_{i=1}^{n} \binom{n}{i}\left(\frac{F_{2m}}{(-1)^{m+1}F_m}\right)^i F_{mi} = \left(\frac{F_m}{(-1)^{m+1}F_m}\right)^n F_{2m}$$

$$\sum_{i=1}^{n} (-1)^{(m+1)i}\binom{n}{i}\left(\frac{F_m L_m}{F_m}\right)^i F_{mi} = (-1)^{(m+1)n} F_{2mn} .$$

Hence

$$F_{2mn} = \begin{cases} \displaystyle\sum_{i=1}^{n} \binom{n}{i} L_m^i F_{mi}, & \text{if } m \text{ is odd} \\ \displaystyle\sum_{i=1}^{n} (-1)^{n+i}\binom{n}{i} L_m^i F_{mi}, & \text{if } m \text{ is even} \end{cases}$$

Also solved by M. Shannon, B. Giuli, and M. Yoder.

HIGHER BRACKET

H-167 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$S_k = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+k}} .$$

Show that, for $k \geq 0$,

$$\text{(A)} \qquad F_{2k+2} S_{2k+2} = k + 1 - \sum_{n=1}^{2k} \frac{k - [\frac{1}{2}(n - 1)]}{F_n F_{n+2}} ,$$

(B) $$F_{2k+1} S_{2k+1} = S_1 - k + \sum_{n=0}^{2k+1} \frac{k - \left[\frac{n}{2}\right]}{F_n F_{n+2}} ,$$

where $[a]$ denotes the greatest integer function.

Special cases of (A) and (B) have been proved by Brother Alfred Brousseau, "Summation of Infinite Fibonacci Series," Fibonacci Quarterly, Vol. 7, No. 2, April, 1969, pp. 143-168.

Solution by the Proposer.

1. Proof of (A). It follows from the identity

$$F_{n+2k} F_{2k+2} - F_{n+2k+2} F_{2k} = F_n$$

that

$$F_{2k+2} S_{2k+2} - F_{2k} S_{2k} = F_{2k+2} \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2k+2}} - F_{2k} \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2k}}$$

$$= \sum_{n=1}^{\infty} \frac{F_{n+2k} F_{2k+2} - F_{n+2k+2} F_{2k}}{F_n F_{n+2k} F_{n+2k+2}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{F_{n+2k} F_{n+2k+2}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} - \sum_{n=1}^{2k} \frac{1}{F_n F_{n+2}} .$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} = 1 ,$$

we get

$$F_{2k+2}S_{2k+2} - F_{2k}S_{2k} = 1 - \sum_{n=1}^{2k} \frac{1}{F_n F_{n+2}} \qquad (k \geq 0) .$$

Then, by addition,

$$F_{2k+2}S_{2k+2} = k + 1 - \sum_{j=1}^{k} \sum_{n=1}^{2j} \frac{1}{F_n F_{n+2}}$$

$$= k + 1 - \sum_{n=1}^{2k} \frac{1}{F_n F_{n+2}} \sum_{n \le 2j \le 2k} 1 .$$

The inner sum is equal to

$$\sum_{\frac{n}{2} \le j \le k} 1 = k - \sum_{1 \le j < \frac{n}{2}} 1 = k - [\tfrac{1}{2}(n - 1)] .$$

Therefore

$$F_{2k+2}S_{2k+2} = k + 1 - \sum_{n=1}^{2k} \frac{k - [\frac{1}{2}(n - 1)]}{F_n F_{n+2}} .$$

This evidently proves (A).

2. Proof of (B). It follows from the identity

$$F_{n+2k+1}F_{2k-1} - F_{n+2k-1}F_{2k+1} = F_n$$

that

$$F_{2k+1} S_{2k+1} - F_{2k-1} S_{2k-1} = F_{2k+1} \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2k+1}} - F_{2k-1} \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2k-1}}$$

$$= \sum_{n=1}^{\infty} \frac{F_{2k+1} F_{n+2k-1} - F_{2k-1} F_{n+2k+1}}{F_n F_{n+2k-1} F_{n+2k+1}}$$

$$= - \sum_{n=1}^{\infty} \frac{1}{F_{n+2k-1} F_{n+2k+1}}$$

$$= - \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} + \sum_{n=1}^{2k-1} \frac{1}{F_n F_{n+2}} ,$$

so that

$$F_{2k+1} S_{2k+1} - F_{2k-1} S_{2k-1} = -1 + \sum_{n=1}^{2k-1} \frac{1}{F_n F_{n+2}} .$$

Then, by addition,

$$F_{2k+1} S_{2k+1} - S_1 = -k + \sum_{j=1}^{k} \sum_{n=1}^{2j-1} \frac{1}{F_n F_{n+2}}$$

$$= -k + \sum_{n=1}^{2k-1} \frac{1}{F_n F_{n+2}} \sum_{n < 2j \leq 2k} 1 .$$

The inner sum is equal to

$$\sum_{\frac{n}{2} < j \leq k} 1 = k - \sum_{j \leq \frac{n}{2}} 1 = k - \left[\frac{n}{2}\right] .$$

Therefore,

$$F_{2k+1}S_{2k+1} = S_1 - k + \sum_{n=1}^{2k-1} \frac{k - \left[\frac{n}{2}\right]}{F_n F_{n+2}} .$$

This proves (B).

Also solved by M. Yoder.

[Continued from page 350.]

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