ADDITIONS TO THE SUMMATION OF RECIPROCAL FIBONACCI AND LUCAS SERIES

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1. In two recent papers [1], [2], Brother U. Alfred Brousseau surveyed the status of the summation of infinite reciprocal Fibonacci series. In this paper, we will add a few summations to those of Brother Brousseau.

We will use the notations Ln and Fn for the n^{th} Lucas and Fibonacci numbers.

2. We have from Bromwich [3]

$$(1) \qquad \frac{x}{1-x^2} + \frac{x^2}{1-x^4} + \cdots + \frac{x^{2^n}}{1-x^{2^{(n+1)}}} = \frac{1}{1-x} - \frac{1}{1-x^{2^{(n+1)}}} .$$

The left-hand expression of (1) can be converted to

$$\frac{\sqrt{5}}{r^{(2^j m)} - r^{(-2^j m)}}$$

if one substitutes $x = r^{2m}$ with r an integer and multiplies by $\sqrt{5}$. We then have

$$(2) \qquad \sum_{j=1}^{n} \frac{1}{F(2^j m)} = \sqrt{5}\left(\frac{1}{r^{(2m)} - 1} - \frac{1}{r^{(m2^{n+1})} - 1}\right) .$$

Clearly (2) gives rise to the infinite formula

$$(3) \qquad \sum_{j=1}^{\infty} \frac{1}{F(2^j m)} = \frac{\sqrt{5}}{r^{2m} - 1} .$$

3. One can easily show

$$
\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \cdots + \frac{2^n}{x^{2^n}+1} = \frac{1}{x-1} - \frac{2^{n+1}}{x^{2^{(n+1)}}-1} \tag{4}
$$

(Jolley [4] gives this formula for the infinite case.) Equation (4) can be converted by the substitution $x = r^{4m}$ into

$$
\sum_{j=0}^{n} \frac{2^j s(2^{j+1}m)}{L(2^{j+1}m)} = \frac{1}{r^{4m}-1} - \frac{2^{n+1}}{r^{(2^n m)}-1} \tag{5}
$$

Since the final term of (5) goes to zero as $n \to \infty$, Eq. (5) gives rise to

$$
\sum_{j=0}^{\infty} \frac{2^j s(2^{j+1}m)}{L(2^{j+1}m)} = \frac{1}{r^{4m}-1} \tag{6}
$$

4. The author has not found the following summation formula in the literature:

$$
\sum \frac{x^{(3^n)} + x^{2(3^n)}}{x^{(3^{n+1})}-1} = \frac{1}{x^3-1} - \frac{1}{x^{(3^{n+1})}-1} \tag{7}
$$

If in (7) we set $x = r^{(4m+2)}$, we obtain

$$
\sum_{j=0}^{n} \frac{F(2m+1)3^j}{L(2m+1)3^{j+1}} = \frac{1}{5}\,\frac{1}{r^{(12m+6)}-1} - \frac{1}{r^{(4m+2)3^{n+1}}-1} \tag{8}
$$

while if in (7) we set $x = r^{(4m)}$,

$$\sum_{j=0}^{n} \frac{F(2m3^j)}{L(2m3^j)} = 5 \; \frac{1}{r^{12m} - 1} - \frac{1}{r^{(12m\,3^{n+1})} - 1} \; . \tag{9}$$

4. Formula (7) suggests the following generalization:

$$\sum_{i=1}^{n} \left\{ \frac{\sum_{j=1}^{k-1} x^{(jk^{j-1})}}{1 - x^{(k^i)}} \right\} = \sum_{i=1}^{n} \frac{x^{(k^i - 1)} - x^{(k^i)}}{\left[1 - x^{(k^i)}\right]\left[1 - x^{(k^i - 1)}\right]} = \frac{1}{1 - x} - \frac{1}{1 - x^{(k^m)}} \tag{10}$$

In (10), if $x = r^{(4n)}$,

$$\sum_{i=1}^{n} \frac{F(2n(k^i - k^{i-1}))}{F(2nk^i)F(2nk^{i-1})} = \sqrt{5}\left[\frac{1}{r^{4n} - 1} - \frac{1}{x^{4nk^m} - 1}\right] \tag{11}$$

In (10) with k odd and $x = r^{(4n+2)}$, we have

$$\sum_{i=1}^{m} \frac{L\,((2n+1)(k^i - k^{i-1}))}{L\{(2n+1)k^i\}L\{(2n+1)k^{i-1}\}} = \frac{1}{r^{(4n+2)} - 1} - \frac{1}{r^{(4n+2)k^m} - 1} \tag{12}$$

Both (11) and (12) become infinite in an obvious way.

REFERENCES

1. Brother Alfred Brousseau, "Summation of Infinite Fibonacci Series," The Fibonacci Quarterly, Vol. 7, No. 2, pp. 143-168.

[Continued on page 412.]