EQUAL PRODUCTS OF GENERALIZED BINOMIAL COEFFICIENTS

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In a letter dated 24 March 1970, Professor V. E. Hoggatt, Jr., has communicated to me the following interesting result: "Choose a binomial coefficient $\binom{n}{k}$ inside Pascal's triangle. There are six bordering terms of Pascal's triangle surrounding $\binom{n}{k}$. The product of all six is a perfect square." As he notes, the theorem is also true for the generalized binomial coefficients $\left\{ {n \atop k} \right\}$ discussed in [1]. In a later communication (22 April 1970), Hoggatt has noted that a corresponding extension to multinomial coefficients holds true. (See [2].)

We may arrange the six binomial coefficients as follows:

$$(1) \qquad \begin{array}{ccccc} & \left\{ {n-1 \atop k-1} \right\} & & \left\{ {n-1 \atop k} \right\} & \\ \left\{ {n \atop k-1} \right\} & & \left\{ {n \atop k} \right\} & & \left\{ {n \atop k+1} \right\} \\ & \left\{ {n+1 \atop k} \right\} & & \left\{ {n+1 \atop k+1} \right\} & \end{array}$$

Here the braces denote the generalized binomial coefficients studied in [1] and defined by

$$(2) \qquad \left\{ {n \atop k} \right\} = \frac{[n]!}{[k]!\,[n-k]!} ,$$

with the generalized factorials given by

$$[n]! = A_n A_{n-1} \cdots A_2 A_1, \qquad [0]! = 1 ,$$

where $\{A_1, A_2, \cdots\}$ is an arbitrary sequence except that $A_i \neq 0$. In the present paper, we shall abbreviate the factorial notation and agree to write (n) instead of $[n]!$.

Now it is easily seen that

$$\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}\begin{Bmatrix} n-1 \\ k \end{Bmatrix}\begin{Bmatrix} n \\ k-1 \end{Bmatrix}\begin{Bmatrix} n \\ k+1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k \end{Bmatrix}\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \tag{3}$$
$$= \left(\frac{(n-1)(n)(n+1)}{(k-1)(k)(k+1)(n-1-k)(n-k)(n+1-k)}\right)^2$$

so that the hexagon theorem is indeed true in general.

Moreover, this is true because in fact two products are equal:

$$\begin{Bmatrix} n-1 \\ k \end{Bmatrix}\begin{Bmatrix} n \\ k-1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}\begin{Bmatrix} n \\ k+1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k \end{Bmatrix} \tag{4}$$

The arrangement of these terms in the original hexagon suggests a Star of David, and we will refer to this form of the theorem as the Star of David property. This property motivates the following paper.

Instead of searching for squares in the general Pascal triangle, we will look for equal products of generalized coefficients. The first such problem which we solve is to find equal products of <u>five</u> binomial coefficients, just as Hoggatt's Star of David property gives such a result for equal products of <u>three</u> binomial coefficients.

First of all, however, we ought to examine into the question of whether there are any other equal products of three. To keep the problem within reasonable bounds we will consider only what happens when we make all six possible permutations of the lower indices $k-1$, k, $k+1$ in a product such as that in (4). The six possible products of three binomial coefficients yield the relation (4) and the remaining set of four products are in general unequal. For example,

$$\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix}\begin{Bmatrix} n \\ k \end{Bmatrix}\begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} = \frac{(n-1)(n)(n+1)}{(k-1)(n-k)(k)(n-k)(k+1)(n-k)}$$

and

$$\begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix}\begin{Bmatrix} n \\ k-1 \end{Bmatrix}\begin{Bmatrix} n+1 \\ k \end{Bmatrix} = \frac{(n-1)(n)(n+1)}{(k+1)(n-2-k)(k-1)(n+1-k)(k)(n+1-k)},$$

which are two different things. We should remark that the simplest possible case of equal products

$$(5) \qquad \left\{\begin{matrix} n + a \\ k + c \end{matrix}\right\}\left\{\begin{matrix} n + b \\ k + d \end{matrix}\right\} = \left\{\begin{matrix} n + a \\ k + d \end{matrix}\right\}\left\{\begin{matrix} n + b \\ k + c \end{matrix}\right\}$$

has only the trivial solutions $a = b$ or $c = d$ or both.

Demanding relation (5) is something rather different from the knowledge that

$$\left\{\begin{matrix} n \\ k \end{matrix}\right\}\left\{\begin{matrix} k \\ j \end{matrix}\right\} = \left\{\begin{matrix} n \\ j \end{matrix}\right\}\left\{\begin{matrix} n - j \\ k - j \end{matrix}\right\},$$

a true identity, because we are concerned solely with permutations of the lower or upper indices.

To go ahead with the situation for a product of five coefficients, we note first that it is not necessary to enumerate all possible products which can be written. It will be sufficient for our purposes to see first of all in how many ways the numbers $k - 2$, $k - 1$, k, $k + 1$, $k + 2$ may be added to the numbers $n - k - 2$, $n - k - 1$, $n - k$, $n - k + 1$, $n - k + 2$ so as to yield some or all of the numbers $n - 2$, $n - 1$, n, $n + 1$, $n + 2$. Now, $k - 2$ may be paired with $n - k$, $n - k + 1$, or $n - k + 2$ only, unless we wish to admit elements such as $n - 3$ or $n + 3$. Our paper will exclude consideration of any numbers in the upper index position other than $n - 2, \cdots, n + 2$.

A list of possible pairings can be written as follows:

$$(6) \qquad \begin{array}{l|l} k - 2 & n - k,\ n - k + 1,\ n - k + 2 \\ k - 1 & n - k - 1,\ n - k,\ n - k + 1,\ n - k + 2 \\ k & n - k - 2,\ n - k - 1,\ n - k,\ n - k + 1,\ n - k + 2 \\ k + 1 & n - k - 2,\ n - k - 1,\ n - k,\ n - k + 1 \\ k + 2 & n - k - 2,\ n - k - 1,\ n - k \end{array}$$

If we denote the five numbers $n - k - 2$, $n - k - 1$, $n - k$, $n - k + 1$, $n - k + 2$ by, respectively, A, B, C, D, E then we may set up the chart more conveniently as follows:

$$
\begin{array}{l|l}
k-2 & C,\ D,\ E \\
k-1 & B,\ C,\ D,\ E \\
k & A,\ B,\ C,\ D,\ E \\
k+1 & A,\ B,\ C,\ D \\
k+2 & A,\ B,\ C
\end{array}
\tag{7}
$$

and all arrangements necessary to consider then may be found by choosing arrangements of the distinct letters in columns, where one letter only may be chosen from a given row in (7). There appear to be just 31 possible combinations:

$$
\begin{array}{ccccccccccccc}
C & C & C & C & C & C & C & D & D & D & D & D & D \\
B & D & D & E & E & E & E & B & B & C & C & E & E \\
E & E & E & A & B & D & D & E & E & E & E & A & A \\
D & B & A & D & D & A & B & A & C & A & B & B & C \\
A & A & B & B & A & B & A & C & A & B & A & C & B \\
\\
D & D & D & D & E & E & E & E & E & E & E & E & E \\
E & E & E & E & D & D & D & D & D & D & C & C & C \\
B & B & C & C & A & A & B & B & C & C & D & D & B \\
A & C & A & B & B & C & A & C & A & B & A & B & D \\
C & A & B & A & C & B & C & A & B & A & B & A & A \\
\\
E & E & E & E & E \\
C & B & B & B & B \\
A & D & D & A & C \\
D & A & C & D & D \\
B & C & A & C & A
\end{array}
\tag{8}
$$

They give a remarkable collection of identities. First of all, there are six combinations that yield the desired $n-2$, $n-1$, n, $n+1$, $n+2$: CEBDA, CDEAB, DBECA, DEABC, EBDAC, and ECABD. The resulting generalized binomial coefficient product identities are:

$$
\left\{
\begin{aligned}
& \left\{\begin{matrix} n-2 \\ k-2 \end{matrix}\right\} \left\{\begin{matrix} n+1 \\ k-1 \end{matrix}\right\} \left\{\begin{matrix} n-1 \\ k \end{matrix}\right\} \left\{\begin{matrix} n+2 \\ k+1 \end{matrix}\right\} \left\{\begin{matrix} n \\ k+2 \end{matrix}\right\} \\
&= \left\{\begin{matrix} n-2 \\ k-2 \end{matrix}\right\} \left\{\begin{matrix} n \\ k-1 \end{matrix}\right\} \left\{\begin{matrix} n+2 \\ k \end{matrix}\right\} \left\{\begin{matrix} n-1 \\ k+1 \end{matrix}\right\} \left\{\begin{matrix} n+1 \\ k+2 \end{matrix}\right\} \\
&= \left\{\begin{matrix} n-1 \\ k-2 \end{matrix}\right\} \left\{\begin{matrix} n-2 \\ k-1 \end{matrix}\right\} \left\{\begin{matrix} n+2 \\ k \end{matrix}\right\} \left\{\begin{matrix} n+1 \\ k+1 \end{matrix}\right\} \left\{\begin{matrix} n \\ k+2 \end{matrix}\right\}
\end{aligned}
\right.
\tag{12}
$$

$$\begin{aligned} &= \left\{ {n-1 \atop k-2} \right\} \left\{ {n+1 \atop k-1} \right\} \left\{ {n-2 \atop k} \right\} \left\{ {n \atop k+1} \right\} \left\{ {n+2 \atop k+2} \right\} \\ &= \left\{ {n \atop k-2} \right\} \left\{ {n-2 \atop k-1} \right\} \left\{ {n+1 \atop k} \right\} \left\{ {n-1 \atop k+1} \right\} \left\{ {n+2 \atop k+2} \right\} \\ &= \left\{ {n \atop k-2} \right\} \left\{ {n-1 \atop k-1} \right\} \left\{ {n-2 \atop k} \right\} \left\{ {n+2 \atop k+1} \right\} \left\{ {n+1 \atop k+2} \right\} . \end{aligned}$$

If we next equate these products in pairs, we find that a common factor cancels out in a number of cases, so that we obtain three different pairs of equal products of five coefficients:

(10)
$$\begin{aligned} &\left\{ {n-2 \atop k-2} \right\} \left\{ {n+1 \atop k-1} \right\} \left\{ {n-1 \atop k} \right\} \left\{ {n+2 \atop k+1} \right\} \left\{ {n \atop k+2} \right\} \\ &= \left\{ {n \atop k-2} \right\} \left\{ {n-2 \atop k-1} \right\} \left\{ {n+1 \atop k} \right\} \left\{ {n-1 \atop k+1} \right\} \left\{ {n+2 \atop k+2} \right\} \end{aligned}$$

(11)
$$\begin{aligned} &\left\{ {n-2 \atop k-2} \right\} \left\{ {n \atop k-1} \right\} \left\{ {n+2 \atop k} \right\} \left\{ {n-1 \atop k+1} \right\} \left\{ {n+1 \atop k+2} \right\} \\ &= \left\{ {n-1 \atop k-2} \right\} \left\{ {n+1 \atop k-1} \right\} \left\{ {n-2 \atop k} \right\} \left\{ {n \atop k+1} \right\} \left\{ {n+2 \atop k+2} \right\} \end{aligned}$$

(12)
$$\begin{aligned} &\left\{ {n-1 \atop k-2} \right\} \left\{ {n-2 \atop k-1} \right\} \left\{ {n+2 \atop k} \right\} \left\{ {n+1 \atop k+1} \right\} \left\{ {n \atop k+2} \right\} \\ &= \left\{ {n \atop k-2} \right\} \left\{ {n-1 \atop k-1} \right\} \left\{ {n-2 \atop k} \right\} \left\{ {n+2 \atop k+1} \right\} \left\{ {n+1 \atop k+2} \right\} \end{aligned}$$

These identities are the natural extension of the Star of David property (4).

Of those cases in (9) where a common factor cancels out, we appear to get twelve equal products of four binomial coefficients:

(13)
$$\left\{ {n+1 \atop k-1} \right\} \left\{ {n-1 \atop k} \right\} \left\{ {n+2 \atop k+1} \right\} \left\{ {n \atop k+2} \right\} = \left\{ {n \atop k-1} \right\} \left\{ {n+2 \atop k} \right\} \left\{ {n-1 \atop k+1} \right\} \left\{ {n+1 \atop k+2} \right\}$$

(14)
$$\left\{ {n-2 \atop k-2} \right\} \left\{ {n+1 \atop k-1} \right\} \left\{ {n-1 \atop k} \right\} \left\{ {n+2 \atop k+1} \right\} = \left\{ {n-1 \atop k-2} \right\} \left\{ {n-2 \atop k-1} \right\} \left\{ {n+2 \atop k} \right\} \left\{ {n+1 \atop k+1} \right\}$$

(15)

$$\genfrac\{\}{0pt}{}{n-2}{k-2}\genfrac\{\}{0pt}{}{n-1}{k}\genfrac\{\}{0pt}{}{n+2}{k+1}\genfrac\{\}{0pt}{}{n}{k+2} = \genfrac\{\}{0pt}{}{n-1}{k-2}\genfrac\{\}{0pt}{}{n-2}{k}\genfrac\{\}{0pt}{}{n}{k+1}\genfrac\{\}{0pt}{}{n+2}{k+2}$$

(16)

$$\genfrac\{\}{0pt}{}{n-2}{k-2}\genfrac\{\}{0pt}{}{n+1}{k-1}\genfrac\{\}{0pt}{}{n-1}{k}\genfrac\{\}{0pt}{}{n}{k+2} = \genfrac\{\}{0pt}{}{n}{k-2}\genfrac\{\}{0pt}{}{n-1}{k-1}\genfrac\{\}{0pt}{}{n-2}{k}\genfrac\{\}{0pt}{}{n+1}{k+2}$$

(17)

$$\genfrac\{\}{0pt}{}{n-2}{k-2}\genfrac\{\}{0pt}{}{n}{k-1}\genfrac\{\}{0pt}{}{n-1}{k+1}\genfrac\{\}{0pt}{}{n+1}{k+2} = \genfrac\{\}{0pt}{}{n-1}{k-2}\genfrac\{\}{0pt}{}{n-2}{k-1}\genfrac\{\}{0pt}{}{n+1}{k+1}\genfrac\{\}{0pt}{}{n}{k+2}$$

(18)

$$\genfrac\{\}{0pt}{}{n-2}{k-2}\genfrac\{\}{0pt}{}{n}{k-1}\genfrac\{\}{0pt}{}{n+2}{k}\genfrac\{\}{0pt}{}{n+1}{k+2} = \genfrac\{\}{0pt}{}{n}{k-2}\genfrac\{\}{0pt}{}{n-2}{k-1}\genfrac\{\}{0pt}{}{n+1}{k}\genfrac\{\}{0pt}{}{n+2}{k+2}$$

(19

$$\genfrac\{\}{0pt}{}{n-2}{k-2}\genfrac\{\}{0pt}{}{n}{k-1}\genfrac\{\}{0pt}{}{n+2}{k}\genfrac\{\}{0pt}{}{n-1}{k+1} = \genfrac\{\}{0pt}{}{n}{k-2}\genfrac\{\}{0pt}{}{n-1}{k-1}\genfrac\{\}{0pt}{}{n-2}{k}\genfrac\{\}{0pt}{}{n+2}{k+1}$$

(20)

$$\genfrac\{\}{0pt}{}{n-2}{k-1}\genfrac\{\}{0pt}{}{n+2}{k}\genfrac\{\}{0pt}{}{n+1}{k+1}\genfrac\{\}{0pt}{}{n}{k+2} = \genfrac\{\}{0pt}{}{n+1}{k-1}\genfrac\{\}{0pt}{}{n-2}{k}\genfrac\{\}{0pt}{}{n}{k+1}\genfrac\{\}{0pt}{}{n+2}{k+2}$$

(21)

$$\genfrac\{\}{0pt}{}{n-1}{k-2}\genfrac\{\}{0pt}{}{n+2}{k}\genfrac\{\}{0pt}{}{n+1}{k+1}\genfrac\{\}{0pt}{}{n}{k+2} = \genfrac\{\}{0pt}{}{n}{k-2}\genfrac\{\}{0pt}{}{n+1}{k}\genfrac\{\}{0pt}{}{n-1}{k+1}\genfrac\{\}{0pt}{}{n+2}{k+2}$$

(22)

$$\genfrac\{\}{0pt}{}{n-1}{k-2}\genfrac\{\}{0pt}{}{n+1}{k-1}\genfrac\{\}{0pt}{}{n-2}{k}\genfrac\{\}{0pt}{}{n}{k+1} = \genfrac\{\}{0pt}{}{n}{k-2}\genfrac\{\}{0pt}{}{n-2}{k-1}\genfrac\{\}{0pt}{}{n+1}{k}\genfrac\{\}{0pt}{}{n-1}{k+1}$$

(23)

$$\genfrac\{\}{0pt}{}{n-1}{k-2}\genfrac\{\}{0pt}{}{n+1}{k-1}\genfrac\{\}{0pt}{}{n}{k+1}\genfrac\{\}{0pt}{}{n+2}{k+2} = \genfrac\{\}{0pt}{}{n}{k-2}\genfrac\{\}{0pt}{}{n-1}{k-1}\genfrac\{\}{0pt}{}{n+2}{k+1}\genfrac\{\}{0pt}{}{n+1}{k+2}$$

(24)

$$\genfrac\{\}{0pt}{}{n-2}{k-1}\genfrac\{\}{0pt}{}{n+1}{k}\genfrac\{\}{0pt}{}{n-1}{k+1}\genfrac\{\}{0pt}{}{n+2}{k+2} = \genfrac\{\}{0pt}{}{n-1}{k-1}\genfrac\{\}{0pt}{}{n-2}{k}\genfrac\{\}{0pt}{}{n+2}{k+1}\genfrac\{\}{0pt}{}{n+1}{k+2}$$

But these twelve are not all distinct. In relation (13) replace k by k - 1 and n by n - 1. This shows that (13) is equivalent to (22). Similarly, (14) and (24) are equivalent and (17) and (23) are equivalent. Thus we obtain nine distinct relations. Of these, only the first, relation (13), has consecutive integers in both the upper and lower index positions, and is thereby an elegant companion to (4). It is an octagonal equivalent of the original Star of David property:

(25)

$$\begin{array}{ccccc}
 & \left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\} & \left\{\begin{matrix} n-1 \\ k \end{matrix}\right\} & \left\{\begin{matrix} n-1 \\ k+1 \end{matrix}\right\} & \\
\left\{\begin{matrix} n \\ k-1 \end{matrix}\right\} & \left\{\begin{matrix} n \\ k \end{matrix}\right\} & \left\{\begin{matrix} n \\ k+1 \end{matrix}\right\} & \left\{\begin{matrix} n \\ k+2 \end{matrix}\right\} & \\
\left\{\begin{matrix} n+1 \\ k-1 \end{matrix}\right\} & \left\{\begin{matrix} n+1 \\ k \end{matrix}\right\} & \left\{\begin{matrix} n+1 \\ k+1 \end{matrix}\right\} & \left\{\begin{matrix} n+1 \\ k+2 \end{matrix}\right\} & \\
 & \left\{\begin{matrix} n+2 \\ k \end{matrix}\right\} & \left\{\begin{matrix} n+2 \\ k+1 \end{matrix}\right\} & \left\{\begin{matrix} n+2 \\ k+2 \end{matrix}\right\} &
\end{array}$$

We return next to the 31 permutations in (8). There are 25 of these which yield products having some repetitions among the numbers n - 2, n - 1, n, n + 1, n + 2. It is worthwhile to explore these. Three of these stand alone: CEADB, DBEAC, and EDCBA. Three pairs give equal products of five coefficients: CBEDA and EBADC; DCEAB and DEBAC; CEDAB and DEACB. Four trios give inequalities: CDEBA, EBCDA, EDABC; DCEBA, ECBDA, EDBAC; DEBCA, DECAB, ECDAB; CEDBA, EBDCA, EDACB. Finally, there is a set of four equalities of products: DECBA, ECDBA, EDCAB, EDBCA. Exploring all the possible pairings, case-by-case, we find first of all three sets of equal products of five coefficients, as follows:

(26)

$$\left\{\begin{matrix} n-1 \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k \end{matrix}\right\}\left\{\begin{matrix} n \\ k+1 \end{matrix}\right\}\left\{\begin{matrix} n \\ k+2 \end{matrix}\right\} = \left\{\begin{matrix} n \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k+1 \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k+2 \end{matrix}\right\}$$

(27)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} ,$$

(28)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} ,$$

and these form interesting geometric patterns when marked in the Pascal triangle. The left and right members in each identity are symmetrical with respect to $\begin{Bmatrix} n \\ k \end{Bmatrix}$ as a central point.

Next, we obtain five sets of equal products of four coefficients:

(29)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} .$$

(30)

$$\begin{Bmatrix} n-1 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-1 \\ k \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+1 \end{Bmatrix} ,$$

(31)

$$\begin{Bmatrix} n-1 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n-1 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+1 \\ k+2 \end{Bmatrix} ,$$

(32)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} ,$$

(33)

$$\begin{Bmatrix} n-2 \\ k-1 \end{Bmatrix} \begin{Bmatrix} n \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+1 \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-1 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+1 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix} .$$

The remainder of the relations found are equal products of three coefficients. The most interesting of these results from equating the permutations CBEDA and EBADC:

(34)

$$\begin{Bmatrix} n-2 \\ k-2 \end{Bmatrix} \begin{Bmatrix} n+2 \\ k \end{Bmatrix} \begin{Bmatrix} n \\ k+2 \end{Bmatrix} = \begin{Bmatrix} n \\ k-2 \end{Bmatrix} \begin{Bmatrix} n-2 \\ k \end{Bmatrix} \begin{Bmatrix} n+2 \\ k+2 \end{Bmatrix} .$$

This is an extension of Hoggatt's original Star of David, and within the Pascal triangle it forms a Star of David with each point moved out one unit further in each direction. What is more, it is easily verified that we have a quite general Star of David formula:

(35)
$$\left\{\begin{matrix} n-a \\ k-a \end{matrix}\right\}\left\{\begin{matrix} n+a \\ k \end{matrix}\right\}\left\{\begin{matrix} n \\ k+a \end{matrix}\right\} = \left\{\begin{matrix} n \\ k-a \end{matrix}\right\}\left\{\begin{matrix} n-a \\ k \end{matrix}\right\}\left\{\begin{matrix} n+a \\ k+a \end{matrix}\right\},$$

where a is an arbitrary integer. Some similar extensions of other relations developed in this paper are possible. It should also be possible to find multinomial extensions.

Relation (34) also follows upon equating permutations CDEBA and EDABC.

Relations equivalent to Hoggatt's original formula are obtained in five cases. Finally, there are six remaining cases:

(36)
$$\left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k \end{matrix}\right\}\left\{\begin{matrix} n+1 \\ k+2 \end{matrix}\right\} = \left\{\begin{matrix} n+1 \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k+2 \end{matrix}\right\}$$

(37)
$$\left\{\begin{matrix} n-2 \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n+1 \\ k \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k+1 \end{matrix}\right\} = \left\{\begin{matrix} n-1 \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n-2 \\ k \end{matrix}\right\}\left\{\begin{matrix} n+1 \\ k+1 \end{matrix}\right\}$$

(38)
$$\left\{\begin{matrix} n-1 \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k \end{matrix}\right\}\left\{\begin{matrix} n \\ k+1 \end{matrix}\right\} = \left\{\begin{matrix} n \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k+1 \end{matrix}\right\}$$

(39)
$$\left\{\begin{matrix} n-1 \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k+1 \end{matrix}\right\}\left\{\begin{matrix} n \\ k+2 \end{matrix}\right\} = \left\{\begin{matrix} n \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n-1 \\ k+1 \end{matrix}\right\}\left\{\begin{matrix} n+2 \\ k+2 \end{matrix}\right\}$$

(40)
$$\left\{\begin{matrix} n-2 \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n+1 \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n \\ k+1 \end{matrix}\right\} = \left\{\begin{matrix} n \\ k-2 \end{matrix}\right\}\left\{\begin{matrix} n-2 \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n+1 \\ k+1 \end{matrix}\right\}$$

(41)
$$\left\{\begin{matrix} n-2 \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n+1 \\ k \end{matrix}\right\}\left\{\begin{matrix} n \\ k+2 \end{matrix}\right\} = \left\{\begin{matrix} n \\ k-1 \end{matrix}\right\}\left\{\begin{matrix} n-2 \\ k \end{matrix}\right\}\left\{\begin{matrix} n+1 \\ k+2 \end{matrix}\right\}.$$

These offer various geometric patterns because they do not involve consecutive integers for the upper and lower indices. As a matter of fact, they all represent Star of David patterns, rotated differently than the original pattern.

Each of the formulas (36) — (41) represents a Star with two points in common with the original Star. From these relations, by means of the substitutions $k+1$ for k, or $a+1$ for n, etc., it is easy to see that relations (36) and (40) are the same, and relations (37) and (39) are the same. The others are distinct from each other and from these. The result is that relations (36), (37), (38), and (41) are the four distinct relations given. One can easily find, as we did in the case for products of five coefficients, whether there are any other distinct such relations.

It would seem to be possible to program the entire procedure for a modern digital computer, which could tirelessly check out all possible cases, and this would make it very easy to tabulate all possible equal products of binomial coefficients within any specified range of parameters. A program could evidently be written along the lines of the procedure used here. Some results, such as formula (35), would not be immediately evident to a computer program, but even here a computer can be programmed to look for certain patterns.

Finally, it would be interesting to find out whether any of the products of the type studied here could be studied in the context of generating functions, as coefficients in power series.

REFERENCES

1. H. W. Gould, "The Bracket Function and Fontene-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients," Fibonacci Quarterly, Vol. 7 (1969), Feb., No. 1, pp. 23-40, 35.
2. V. E. Hoggatt, Jr., and G. L. Alexanderson, "A Property of Multinomial Coefficients," Fibonacci Quarterly, Vol. 9, No. 4, pp. 351-356.